

Quasi Conjunction, Quasi Disjunction, T-norms and T-conorms: Probabilistic Aspects

Angelo Gilio^a, Giuseppe Sanfilippo^{b,*}

^a*Dipartimento di Scienze di Base e Applicate per l'Ingegneria, University of Rome "La Sapienza", Via Antonio Scarpa 16, 00161 Roma, Italy*

^b*Dipartimento di Matematica e Informatica, University of Palermo, Via Archirafi 34, 90123 Palermo, Italy*

Abstract

We make a probabilistic analysis related to some inference rules which play an important role in nonmonotonic reasoning. In a coherence-based setting, we study the extensions of a probability assessment defined on n conditional events to their quasi conjunction, and by exploiting duality, to their quasi disjunction. The lower and upper bounds coincide with some well known t-norms and t-conorms: minimum, product, Lukasiewicz, and Hamacher t-norms and their dual t-conorms. On this basis we obtain Quasi And and Quasi Or rules. These are rules for which any finite family of conditional events p-entails the associated quasi conjunction and quasi disjunction. We examine some cases of logical dependencies, and we study the relations among coherence, inclusion for conditional events, and p-entailment. We also consider the Or rule, where quasi conjunction and quasi disjunction of premises coincide with the conclusion. We analyze further aspects of quasi conjunction and quasi disjunction, by computing probabilistic bounds on premises from bounds on conclusions. Finally, we consider biconditional events, and we introduce the notion of an n -conditional event. Then we give a probabilistic interpretation for a generalized Loop rule. In an appendix we provide explicit expressions for the Hamacher t-norm and t-conorm in the unitary hypercube.

Keywords: coherence, lower/upper probability bounds, quasi conjunction/disjunction, t-norms/conorms, Goodman-Nguyen inclusion relation, generalized Loop rule.

1. Introduction

In classical (monotonic) logic, if a conclusion C follows from some premises, then C also follows when the set of premises is enlarged; that is, adding premises never invalidates any conclusions. In contrast, in (nonmonotonic) commonsense reasoning we are typically in a situation of partial knowledge, and a conclusion reached from a set of premises may be retracted when some premises are added. Nonmonotonic reasoning is a relevant topic in the field of artificial intelligence, and has been studied in literature by many symbolic and numerical formalisms (see, e.g. [6, 8, 9, 22, 54]). A remarkable theory related to nonmonotonic reasoning has been proposed by Adams in his probabilistic logic of conditionals ([1]). We recall that the approach of Adams can be developed with full generality by exploiting coherence-based probabilistic reasoning ([26]). In the setting of coherence conditional probabilities can be directly assigned, and *zero probabilities for conditioning events* can be properly managed (see, e.g. [7, 10, 11, 12, 18, 35, 40, 44, 62]). The coherence-based approach is applied in many fields: statistical analysis, decision theory, probabilistic default reasoning and fuzzy theory. It allows one to manage incomplete probabilistic assignments in a situation of vague or partial knowledge (see, e.g. [13, 14, 15, 17, 19, 37, 55, 56, 64]). A basic notion in the work of Adams is the quasi conjunction of conditionals. This logical operation also plays a relevant role in [22] (see also [6]), where a suitable Quasi And rule is introduced to characterize entailment from a knowledge base. In the present article, besides quasi conjunction, we study by duality the quasi disjunction of conditional events and the

[☆]Both authors contributed equally to this work

^{*}Corresponding author

Email addresses: angelo.gilio@sba.uniroma1.it (Angelo Gilio), giuseppe.sanfilippo@unipa.it (Giuseppe Sanfilippo)

associated Quasi Or rule.

Theoretical tools which play a relevant role in artificial intelligence and fuzzy logic are t-norms and t-conorms. These allow one to extend the Boolean operations of conjunction and disjunction to the setting of multi-valued logics. T-norms (first proposed in [58]) and t-conorms were introduced in [63] and are a subclass of aggregation functions ([46, 47, 48]). They play a basic role in decision theory, information and data fusion, probability theory and risk management.

In this paper we give many insights about probabilistic default reasoning in the setting of coherence, by making a probabilistic analysis of the Quasi And, Quasi Or and Loop inference rules. Some results were already given without proof in [39]. To begin, we recall some basic notions and results regarding coherence, probabilistic default reasoning, and the Hamacher t-norm/t-conorm (Section 2). Then, we show that some well known t-norms and t-conorms appear as lower and upper bounds when we propagate probability assessments on a finite family of conditional events to the associated quasi conjunction. By these bounds we obtain the Quasi And rule. We also consider special cases of logical dependencies associated with the Goodman-Nguyen inclusion relation ([45]) and with the compound probability theorem. Then, we give two results which identify the strict relationship holding among coherence, the Goodman-Nguyen inclusion relation, and p-entailment (Section 3). We deepen a further aspect of the Quasi And rule by determining the probability bounds on the premises from given bounds on the conclusion of the rule (Section 4). By exploiting duality, we give analogous results for the quasi disjunction of conditional events, and we obtain the Quasi Or rule. We also examine the Or rule, and we show that quasi conjunction and quasi disjunction of the premises of this rule both coincide with its conclusion (Section 5). In a similar way, we then enrich the Quasi Or rule by determining the probability bounds on the premises from given bounds on the conclusion of the rule (Section 6). We consider biconditional events, and we introduce the notion of an n -conditional event, by means of which we give a probabilistic semantics to a *generalized Loop rule* (Section 7). Finally, we give some conclusions and perspectives on future work (Section 8). We illustrate notions and results with a table and some figures.

The results given in this work may be useful for the treatment of uncertainty in many applications of statistics and artificial intelligence, in particular for the probabilistic approach to inference rules in nonmonotonic reasoning, for the psychology of uncertain reasoning, and for probabilistic reasoning in the semantic web (see, e.g., [38, 51, 57, 60, 61]).

2. Some Preliminary Notions

In this section we first discuss some basic notions regarding coherence. Then, we recall the notions of p-consistency and p-entailment of Adams ([1]) within the setting of coherence.

2.1. Basic notions on coherence

As in the approach of de Finetti, events represent uncertain facts described by (non ambiguous) logical propositions. An event A is a two-valued logical entity which can be true (T), or false (F). The indicator of A is a two-valued numerical quantity which is 1, or 0, according to whether A is true, or false. We denote by Ω the sure event and by \emptyset the impossible one. We use the same symbols for events and their indicators. Moreover, we denote by $A \wedge B$ (resp., $A \vee B$) the logical intersection, or conjunction (resp., logical union, or disjunction). To simplify notations, in many cases we denote the conjunction between A and B as the product AB . We denote by A^c the negation of A . Of course, the truth values for conjunctions, disjunctions and negations are obtained by applying the propositional calculus. Given any events A and B , we simply write $A \subseteq B$ to denote that A logically implies B , that is $AB^c = \emptyset$, which means that A and B^c cannot be both true. Given n events A_1, \dots, A_n , as $A_i \vee A_i^c = \Omega$, $i = 1, \dots, n$, by expanding the expression $\bigwedge_{i=1}^n (A_i \vee A_i^c)$, we can represent Ω as the disjunction of 2^n logical conjunctions, some of which may be impossible. The remaining ones are the atoms, or constituents, generated by A_1, \dots, A_n . We recall that A_1, \dots, A_n are logically independent when the number of atoms generated by them is 2^n . Of course, in case of some logical dependencies among A_1, \dots, A_n the number of atoms is less than 2^n . For instance, given two logically incompatible events A, B , as $AB = \emptyset$ the atoms are: AB^c, A^cB, A^cB^c . We remark that, to introduce the basic notions, an equivalent approach is that of considering a Boolean algebra \mathcal{B} whose elements are interpreted as events. In this way events would be combined by means of the Boolean operations; then to say that A_1, \dots, A_n are logically independent would mean that the subalgebra generated by them has 2^n atoms. Concerning conditional events, given two events A, B , with $A \neq \emptyset$, in our approach the conditional event $B|A$ is defined as a three-valued logical entity which is true (T), or false (F), or void

(V), according to whether AB is true, or AB^c is true, or A^c is true, respectively. We recall that, agreeing to the betting metaphor, if you assess $P(B|A) = p$, then you are willing to pay an amount p and to receive 1, or 0, or p , according to whether AB is true, or AB^c is true, or A^c is true (bet called off), respectively. Given a real function $P : \mathcal{F} \rightarrow \mathcal{R}$, where \mathcal{F} is an arbitrary family of conditional events, let us consider a subfamily $\mathcal{F}_n = \{E_1|H_1, \dots, E_n|H_n\} \subseteq \mathcal{F}$, and the vector $\mathcal{P}_n = (p_1, \dots, p_n)$, where $p_i = P(E_i|H_i)$, $i = 1, \dots, n$. We denote by \mathcal{H}_n the disjunction $H_1 \vee \dots \vee H_n$. As $E_i H_i \vee E_i^c H_i \vee H_i^c = \Omega$, $i = 1, \dots, n$, by expanding the expression $\bigwedge_{i=1}^n (E_i H_i \vee E_i^c H_i \vee H_i^c)$, we can represent Ω as the disjunction of 3^n logical conjunctions, some of which may be impossible. The remaining ones are the atoms, or constituents, generated by the family \mathcal{F}_n and, of course, are a partition of Ω . We denote by C_1, \dots, C_m the constituents contained in \mathcal{H}_n and (if $\mathcal{H}_n \neq \Omega$) by C_0 the remaining constituent $\mathcal{H}_n^c = H_1^c \dots H_n^c$, so that

$$\mathcal{H}_n = C_1 \vee \dots \vee C_m, \quad \Omega = \mathcal{H}_n^c \vee \mathcal{H}_n = C_0 \vee C_1 \vee \dots \vee C_m, \quad m+1 \leq 3^n.$$

Interpretation with the betting scheme. With the pair $(\mathcal{F}_n, \mathcal{P}_n)$ we associate the random gain $\mathcal{G} = \sum_{i=1}^n s_i H_i (E_i - p_i)$, where s_1, \dots, s_n are n arbitrary real numbers. We observe that \mathcal{G} is the difference between the amount that you receive, $\sum_{i=1}^n s_i (E_i H_i + p_i H_i^c)$, and the amount that you pay, $\sum_{i=1}^n s_i p_i$, and represents the net gain from engaging each transaction $H_i(E_i - p_i)$, the scaling and meaning (buy or sell) of the transaction being specified by the magnitude and the sign of s_i respectively. Let g_h be the value of \mathcal{G} when C_h is true; of course, $g_0 = 0$. Denoting by $G_{\mathcal{H}_n} = \{g_1, \dots, g_m\}$ the set of values of \mathcal{G} restricted to \mathcal{H}_n , we have

Definition 1. The function P defined on \mathcal{F} is said to be *coherent* if and only if, for every integer n , for every finite sub-family $\mathcal{F}_n \subseteq \mathcal{F}$ and for every s_1, \dots, s_n , one has: $\min G_{\mathcal{H}_n} \leq 0 \leq \max G_{\mathcal{H}_n}$.

Notice that the condition $\min G_{\mathcal{H}_n} \leq 0 \leq \max G_{\mathcal{H}_n}$ can be written in two equivalent ways: $\min G_{\mathcal{H}_n} \leq 0$, or $\max G_{\mathcal{H}_n} \geq 0$. As shown by Definition 1, a probability assessment is coherent if and only if, in any finite combination of n bets, it does not happen that the values g_1, \dots, g_m are all positive, or all negative (*no Dutch Book*).

Coherence with penalty criterion. An equivalent notion of coherence for unconditional events and random quantities was introduced by de Finetti ([24, 25, 26]) using the penalty criterion associated with the quadratic scoring rule. Such a penalty criterion has been extended to the case of conditional events in [30]. With the pair $(\mathcal{F}_n, \mathcal{P}_n)$ we associate the loss $\mathcal{L} = \sum_{i=1}^n H_i (E_i - p_i)^2$; we denote by L_h the value of \mathcal{L} if C_h is true. If you specify the assessment \mathcal{P}_n on \mathcal{F}_n as representing your belief's degrees, you are required to pay a penalty L_h when C_h is true. Then, we have

Definition 2. The function P defined on \mathcal{F} is said to be *coherent* if and only if there does not exist an integer n , a finite sub-family $\mathcal{F}_n \subseteq \mathcal{F}$, and an assessment $\mathcal{P}_n^* = (p_1^*, \dots, p_n^*)$ on \mathcal{F}_n such that, for the loss $\mathcal{L}^* = \sum_{i=1}^n H_i (E_i - p_i^*)^2$, associated with $(\mathcal{F}_n, \mathcal{P}_n^*)$, it results $\mathcal{L}^* \leq \mathcal{L}$ and $\mathcal{L}^* \neq \mathcal{L}$; that is $L_h^* \leq L_h$, $h = 1, \dots, m$, with $L_h^* < L_h$ in at least one case.

We can develop a geometrical approach to coherence by associating, with each constituent C_h contained in \mathcal{H}_n , a point $Q_h = (q_{h1}, \dots, q_{hn})$, where $q_{hj} = 1$, or 0, or p_j , according to whether $C_h \subseteq E_j H_j$, or $C_h \subseteq E_j^c H_j$, or $C_h \subseteq H_j^c$. Then, denoting by \mathcal{I} the convex hull of Q_1, \dots, Q_m , the following characterization of coherence w.r.t. penalty criterion can be given ([30, Theorem 4.4], see also [12, 31])

Theorem 1. The function P defined on \mathcal{F} is coherent if and only if, for every finite sub-family $\mathcal{F}_n \subseteq \mathcal{F}$, one has $\mathcal{P}_n \in \mathcal{I}$.

Equivalence between betting scheme and penalty criterion. The betting scheme and the penalty criterion are *equivalent*, as can be proved by the following steps ([34]):

1. The condition $\mathcal{P}_n \in \mathcal{I}$ amounts to solvability of the following system (Σ) in the unknowns $\lambda_1, \dots, \lambda_m$

$$(\Sigma) \quad \begin{cases} \sum_{h=1}^m q_{hj} \lambda_h = p_j, & j = 1, \dots, n; \\ \sum_{h=1}^m \lambda_h = 1, & \lambda_h \geq 0, \quad h = 1, \dots, m. \end{cases}$$

We say that system (Σ) is associated with the pair $(\mathcal{F}_n, \mathcal{P}_n)$.

2. Let $\mathbf{x} = (x_1, \dots, x_m)$, $\mathbf{y} = (y_1, \dots, y_n)^t$ and $A = (a_{ij})$ be, respectively, a row m -vector, a column n -vector and a $m \times n$ -matrix. The vector \mathbf{x} is said *semipositive* if $x_i \geq 0$, $\forall i$, and $x_1 + \dots + x_m > 0$. Then, we have (cf. [28, Theorem 2.9])

Theorem 2. Exactly one of the following alternatives holds.

- (i) the equation $\mathbf{x}A = 0$ has a *semipositive* solution;
- (ii) the inequality $A\mathbf{y} > 0$ has a solution.

3. By choosing $a_{ij} = q_{ij} - p_j$, $\forall i, j$, the solvability of $\mathbf{x}A = 0$ means that $\mathcal{P}_n \in \mathcal{I}$, while the solvability of $A\mathbf{y} > 0$ means that, choosing $s_i = y_i$, $\forall i$, one has $\min G_{\mathcal{H}_n} > 0$. Hence, by applying Theorem 2 with $A = (q_{ij} - p_j)$, we obtain $\max G_{\mathcal{H}_n} \geq 0$ if and only if (Σ) is solvable. In other words, $\max G_{\mathcal{H}_n} \geq 0$ if and only if $\mathcal{P}_n \in \mathcal{I}$. Therefore, Definition 1 and Definition 2 are equivalent.

2.2. Coherence Checking

Given the assessment \mathcal{P}_n on \mathcal{F}_n , let S be the set of solutions $\Lambda = (\lambda_1, \dots, \lambda_m)$ of the system (Σ) . Then, assuming $S \neq \emptyset$, define

$$\begin{aligned}\Phi_j(\Lambda) &= \Phi_j(\lambda_1, \dots, \lambda_m) = \sum_{r: C_r \subseteq H_j} \lambda_r, \quad j = 1, \dots, n; \quad \Lambda \in S; \\ M_j &= \max_{\Lambda \in S} \Phi_j(\Lambda), \quad j = 1, \dots, n; \quad I_0 = \{j : M_j = 0\}.\end{aligned}$$

We observe that, assuming \mathcal{P}_n coherent, each solution $\Lambda = (\lambda_1, \dots, \lambda_m)$ of system (Σ) is a coherent extension of the assessment \mathcal{P}_n on \mathcal{F}_n to the family $\{C_1|\mathcal{H}_n, \dots, C_m|\mathcal{H}_n\}$. Then, by the additive property, the quantity $\Phi_j(\Lambda)$ is the conditional probability $P(H_j|\mathcal{H}_n)$ and the quantity M_j is the upper probability $P^*(H_j|\mathcal{H}_n)$ over all the solutions Λ of system (Σ) . Of course, $j \in I_0$ if and only if $P^*(H_j|\mathcal{H}_n) = 0$. Notice that $I_0 \subset \{1, \dots, n\}$. We denote by $(\mathcal{F}_0, \mathcal{P}_0)$ the pair associated with I_0 . Given the pair $(\mathcal{F}_n, \mathcal{P}_n)$ and a subset $J \subset J_n = \{1, \dots, n\}$, we denote by $(\mathcal{F}_J, \mathcal{P}_J)$ the pair associated with J and by Σ_J the corresponding system. We observe that (Σ_J) is solvable if and only if $\mathcal{P}_J \in \mathcal{I}_J$, where \mathcal{I}_J is the convex hull associated with the pair $(\mathcal{F}_J, \mathcal{P}_J)$. Then, we have ([32, Theorem 3.2]; see also [10, 33])

Theorem 3. Given a probability assessment \mathcal{P}_n on the family \mathcal{F}_n , if the system (Σ) associated with $(\mathcal{F}_n, \mathcal{P}_n)$ is solvable, then for every $J \subset \{1, \dots, n\}$, such that $J \setminus I_0 \neq \emptyset$, the system (Σ_J) associated with $(\mathcal{F}_J, \mathcal{P}_J)$ is solvable too.

The previous result says that the condition $\mathcal{P}_n \in \mathcal{I}$ implies $\mathcal{P}_J \in \mathcal{I}_J$ when $J \setminus I_0 \neq \emptyset$. We observe that, if $\mathcal{P}_n \in \mathcal{I}$, then for every nonempty subset J of $J_n \setminus I_0$ it holds that $J \setminus I_0 = J \neq \emptyset$; hence, by Theorem 1, the subassessment $\mathcal{P}_{J_n \setminus I_0}$ on the subfamily $\mathcal{F}_{J_n \setminus I_0}$ is coherent. In particular, when I_0 is empty, coherence of \mathcal{P}_n amounts to solvability of system (Σ) , that is to condition $\mathcal{P}_n \in \mathcal{I}$. When I_0 is not empty, coherence of \mathcal{P}_n amounts to the validity of both conditions $\mathcal{P}_n \in \mathcal{I}$ and \mathcal{P}_0 coherent, as shown by the result below ([32, Theorem 3.3])

Theorem 4. The assessment \mathcal{P}_n on \mathcal{F}_n is coherent if and only if the following conditions are satisfied: (i) $\mathcal{P}_n \in \mathcal{I}$; (ii) if $I_0 \neq \emptyset$, then \mathcal{P}_0 is coherent.

2.3. Basic notions on probabilistic default reasoning

Given a conditional knowledge base $\mathcal{K}_n = \{H_i \sim E_i, i = 1, 2, \dots, n\}$, we denote by $\mathcal{F}_n = \{E_i|H_i, i = 1, 2, \dots, n\}$ the associated family of conditional events. We give below, in the setting of coherence, synthetic definitions of the notions of p-consistency and p-entailment of Adams, which are related with [8, Theorem 4.5, Theorem 4.9], [41, Theorem 5], [42, Theorem 6].

Definition 3. The knowledge base $\mathcal{K}_n = \{H_i \sim E_i, i = 1, 2, \dots, n\}$ is *p-consistent* if and only if the assessment $(p_1, p_2, \dots, p_n) = (1, 1, \dots, 1)$ on \mathcal{F}_n is coherent.

Definition 4. A p-consistent knowledge base $\mathcal{K}_n = \{H_i \sim E_i, i = 1, \dots, n\}$ *p-entails* the conditional $A \sim B$, denoted $\mathcal{K}_n \Rightarrow_p A \sim B$, if and only if, for every coherent assessment $(p_1, p_2, \dots, p_n, z)$ on $\mathcal{F}_n \cup \{B|A\}$ such that $(p_1, p_2, \dots, p_n) = (1, 1, \dots, 1)$, it holds that $z = 1$.

The previous definitions of p-consistency and p-entailment are equivalent (see [35, Theorem 8], [41, Theorem 5], [42, Theorem 6]) to that ones given in [35].

Remark 1. We say that a family of conditional events \mathcal{F}_n p-entails a conditional event $B|A$ when the associated knowledge base \mathcal{K}_n p-entails the conditional $A \sim B$.

Definition 4 can be generalized to p-entailment of a family (of conditional events) Γ from another family \mathcal{F} in the following way.

Definition 5. Given two p-consistent finite families of conditional events \mathcal{F} and \mathcal{S} , we say that \mathcal{F} p-entails \mathcal{S} if \mathcal{F} p-entails $E|H$ for every $E|H \in \mathcal{S}$.

We remark that, from Definition 4, we trivially have that \mathcal{F} p-entails $E|H$, for every $E|H \in \mathcal{F}$; then, by Definition 5, it immediately follows

$$\mathcal{F} \Rightarrow_p \mathcal{S}, \quad \forall \mathcal{S} \subseteq \mathcal{F}. \quad (1)$$

Probabilistic default reasoning has been studied by many authors (see, e.g., [6, 8, 9, 22, 54]); methods and results based on the maximum entropy principle have been given in [50, 65].

2.4. Hamacher t-norm and t-conorm

We recall that the Hamacher t-norm, with parameter $\lambda = 0$, or Hamacher product, T_0^H is defined as ([49])

$$T_0^H(x, y) = \begin{cases} 0, & (x, y) = (0, 0), \\ \frac{xy}{x+y-xy}, & (x, y) \neq (0, 0). \end{cases} \quad (2)$$

We also recall that the Hamacher t-conorm, with parameter $\lambda = 0$, S_0^H is

$$S_0^H(x, y) = \begin{cases} 1, & (x, y) = (1, 1), \\ \frac{x+y-2xy}{1-xy}, & (x, y) \neq (1, 1). \end{cases} \quad (3)$$

As is well known, t-norms overlap with copulas ([3, 59]); indeed, commutative associative copulas are t-norms and t-norms which satisfy the 1-Lipschitz condition are copulas. We also recall that some well-known families of t-norms receive a different name in the literature when considered as families of copulas. In particular, the Hamacher product is a copula because it satisfies the following necessary and sufficient condition ([3, Theorem 1.4.5]):

Theorem 5. A t-norm T is a copula if and only if it satisfies the Lipschitz condition: $T(x_2, y) - T(x_1, y) \leq x_2 - x_1$, whenever $x_2 \leq x_1$.

Hamacher product is called Ali-Mikhail-Haq copula with parameter 0 ([2, 46, 52, 59]). Further details on t-norms and t-conorms are given in the Appendices.

3. Lower and Upper Bounds for Quasi Conjunction

We recall below the notion of quasi conjunction of conditional events as defined in [1].

Definition 6. Given any events A, H, B, K , with $H \neq \emptyset, K \neq \emptyset$, the quasi conjunction of the conditional events $A|H$ and $B|K$ is the conditional event $C(A|H, B|K) = (AH \vee H^c) \wedge (BK \vee K^c) | (H \vee K)$, or equivalently $C(A|H, B|K) = (AHBK \vee AHK^c \vee H^cBK) | (H \vee K)$.

Table 1 shows the truth-table of the quasi conjunction $C(A|H, B|K)$ and of the quasi disjunction $\mathcal{D}(A|H, B|K)$ (see Section 5). In general, given a family of n conditional events $\mathcal{F}_n = \{E_i|H_i, i = 1, \dots, n\}$, we have

$$C(\mathcal{F}_n) = C(E_1|H_1, \dots, E_n|H_n) = \bigwedge_{i=1}^n (E_i H_i \vee H_i^c) | \left(\bigvee_{i=1}^n H_i \right).$$

Quasi conjunction is associative; that is, for every subset $J \subset \{1, \dots, n\}$, it holds that $C(\mathcal{F}_n) = C(\mathcal{F}_J \cup \mathcal{F}_{\Gamma}) = C[C(\mathcal{F}_J), C(\mathcal{F}_{\Gamma})]$, where $\Gamma = \{1, \dots, n\} \setminus J$. An interesting analysis of many three-valued logics studied in the literature has been given by Ciucci and Dubois in [16]. In such a paper the definition of conjunction satisfies left monotonicity, right monotonicity and conformity with Boolean logic; then the authors show that there are 14 different ways of defining conjunction and only 5 of them (one of which defines quasi conjunction) satisfy commutativity and associativity.

Assuming A, H, B, K logically independent, we have ([36], see also [37]):

(i) the probability assessment (x, y) on $\{A|H, B|K\}$ is coherent for every $(x, y) \in [0, 1]^2$;

(ii) given a coherent assessment (x, y) on $\{A|H, B|K\}$, the extension $\mathcal{P} = (x, y, z)$ on $\mathcal{F} = \{A|H, B|K, C(A|H, B|K)\}$, with $z = P[C(A|H, B|K)]$, is coherent if and only if $z \in [l, u]$, with

$$l = T_L(x, y) = \max(x + y - 1, 0), \quad u = S_0^H(x, y) = \begin{cases} \frac{x+y-2xy}{1-xy}, & (x, y) \neq (1, 1), \\ 1, & (x, y) = (1, 1), \end{cases} \quad (4)$$

where T_L is the Lukasiewicz t-norm (see Appendix A) and S_0^H is the Hamacher t-conorm¹, with parameter $\lambda = 0$. The lower bound T_L for the quasi conjunction is the Fréchet-Hoeffding lower bound; both l and u coincide with the Fréchet-Hoeffding bounds if we consider the *conjunction* of conditional events in the setting of conditional random quantities, as made in [43]. The lower and upper bounds, l, u , of $z = P[C(A|H, B|K)]$ can be obtained by studying the coherence of the assessment $\mathcal{P} = (x, y, z)$, based on the geometrical approach described in Section 2. The constituents generated by the family $\{A|H, B|K, C(A|H, B|K)\}$ and the corresponding points Q_h 's are given in columns 2 and 6 of Table 1. In [36] (see also [37]) the values l, u are computed by observing that the coherence of $\mathcal{P} = (x, y, z)$ simply amounts to the geometrical condition $\mathcal{P} \in \mathcal{I}$, where \mathcal{I} is the convex hull of the points Q_1, Q_2, \dots, Q_8 (associated with the constituents C_1, C_2, \dots, C_8 contained in $H \vee K$). We observe that in this case the convex hull \mathcal{I} does not depend on z . Figure 1 shows, for given values x, y , the convex hull \mathcal{I} and the associated interval $[l, u]$ for $z = P[C(A|H, B|K)]$.

h	C_h	$A H$	$B K$	$C(A H, B K)$	Q_h	$\mathcal{D}(A H, B K)$	Q_h
0	$H^c K^c$	Void	Void	Void	(x, y, z)	Void	(x, y, z)
1	$AHBK$	True	True	True	$(1, 1, 1)$	True	$(1, 1, 1)$
2	AHK^c	True	Void	True	$(1, y, 1)$	True	$(1, y, 1)$
3	$AHB^c K$	True	False	False	$(1, 0, 0)$	True	$(1, 0, 1)$
4	$H^c BK$	Void	True	True	$(x, 1, 1)$	True	$(x, 1, 1)$
5	$H^c B^c K$	Void	False	False	$(x, 0, 0)$	False	$(x, 0, 0)$
6	$A^c HBK$	False	True	False	$(0, 1, 0)$	True	$(0, 1, 1)$
7	$A^c HK^c$	False	Void	False	$(0, y, 0)$	False	$(0, y, 0)$
8	$A^c HB^c K$	False	False	False	$(0, 0, 0)$	False	$(0, 0, 0)$

Table 1: Truth-Table of the quasi conjunction and of the quasi disjunction with the associated points Q_h 's.

Remark 2. Notice that, if the events A, B, H, K were not logically independent, then some constituents C_h 's (at least one) would become impossible and the lower bound l could increase, while the upper bound u could decrease. To examine this aspect we will consider some special cases of logical dependencies.

3.1. The Case $A|H \subseteq B|K$

The notion of logical inclusion among events has been generalized to conditional events by Goodman and Nguyen in [45]. We recall below this generalized notion.

Definition 7. Given two conditional events $A|H$ and $B|K$, we say that $A|H$ implies $B|K$, denoted by $A|H \subseteq B|K$, iff AH true implies BK true and $B^c K$ true implies $A^c H$ true; i.e., iff $AH \subseteq BK$ and $B^c K \subseteq A^c H$.

Remark 3. Denoting by $t(\cdot)$ the truth value function and assuming the order $False < Void < True$, then it can be easily verified that

$$\begin{aligned} A|H \subseteq B|K &\Leftrightarrow AHB^c K = H^c B^c K = AHK^c = \emptyset, \\ A|H \subseteq B|K &\Leftrightarrow t(A|H) \leq t(B|K) \Leftrightarrow t(B^c|K) \leq t(A^c|H) \Leftrightarrow B^c|K \subseteq A^c|H. \end{aligned}$$

Given any conditional events $A|H, B|K$, we denote by Π_x the set of coherent probability assessments x on $A|H$, by Π_y the set of coherent probability assessments y on $B|K$ and by Π the set of coherent probability assessments (x, y) on $\{A|H, B|K\}$; moreover we indicate by $T_{x \leq y}$ the triangle $\{(x, y) \in [0, 1]^2 : x \leq y\}$. We have

¹ The coincidence between the upper bound u and Hamacher t-conorm $S_0^H(x, y)$ was noticed by Didier Dubois.

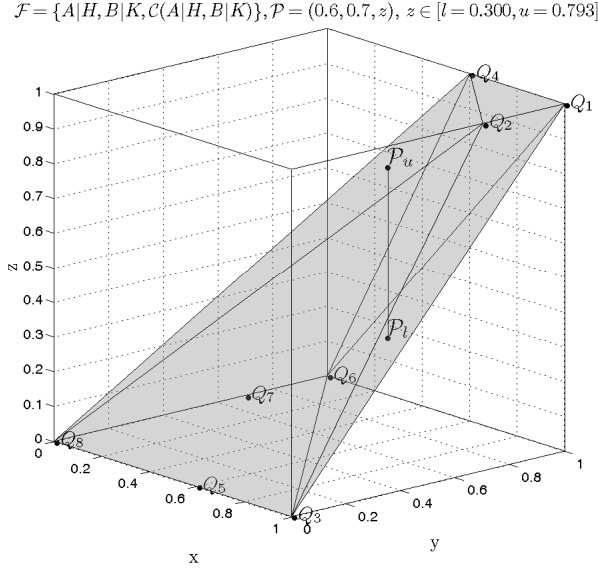


Figure 1: The convex hull I associated with the pair $(\mathcal{F}, \mathcal{P})$ in case of quasi conjunction without logical dependencies. The interval $[l, u]$ for $z = P[C(A|H, B|K)]$ is the range of the third coordinate z of each $\mathcal{P} \in \overline{\mathcal{P}_l \mathcal{P}_u} = \{(x, y, z) : z \in [T_L(x, y), S_0^H(x, y)]\}$. The segment $\overline{\mathcal{P}_l \mathcal{P}_u}$ is the intersection between the segment $\{(x, y, z) : z \in [0, 1]\}$ and the convex hull I .

Theorem 6. Given two conditional events $A|H, B|K$, we have

$$\Pi \subseteq T_{x \leq y} \iff A|H \subseteq B|K, \text{ or } AH = \emptyset, \text{ or } B^c K = \emptyset. \quad (5)$$

Proof. (\Rightarrow) We will prove that

$$A|H \not\subseteq B|K, AH \neq \emptyset, B^c K \neq \emptyset \implies \Pi \not\subseteq T_{x \leq y}. \quad (6)$$

We observe that $AH = \emptyset$ if and only if $\Pi_x = \{0\}$ and that $B^c K = \emptyset$ if and only if $\Pi_y = \{1\}$. Moreover, by Remark 3 it holds

$$A|H \not\subseteq B|K \iff AHB^c K \vee H^c B^c K \vee AHK^c \neq \emptyset.$$

Then, in order to prove formula (6), we distinguish three cases:

(i) $AHB^c K \neq \emptyset$; (ii) $H^c B^c K \neq \emptyset, AH \neq \emptyset$; (iii) $AHK^c \neq \emptyset, B^c K \neq \emptyset$.

In case (i), the assessment $(1, 0)$ on $\{A|H, B|K\}$ is coherent. In case (ii), as $AH \neq \emptyset$ we have $\{1\} \subseteq \Pi_x$; then, the assessment $(1, 0)$ on $\{A|H, B|K\}$ is coherent. In case (iii), as $B^c K \neq \emptyset$ we have $\{0\} \subseteq \Pi_y$; then, the assessment $(1, 0)$ on $\{A|H, B|K\}$ is coherent. Then, in each of the three cases the assessment $(1, 0)$ is coherent and hence $\Pi \not\subseteq T_{x \leq y}$.

(\Leftarrow) We distinguish three cases:

(a) $A|H \subseteq B|K$; (b) $AH = \emptyset$; (c) $B^c K = \emptyset$.

(a) The constituents generated by $\{A|H, B|K\}$ and contained in $H \vee K$ belong to the family:

$$\{AHBK, H^c BK, A^c HBK, A^c HK^c, A^c HB^c K\}.$$

The corresponding points Q_h 's belong to the set $\{(1, 1), (x, 1), (0, 1), (0, y), (0, 0)\}$, which has the triangle $T_{x \leq y}$ as convex hull; hence the convex hull Π of the points Q_h 's is a subset of $T_{x \leq y}$.

(b) Since $\Pi_x = \{0\}$ it follows that $\Pi \subseteq \{(0, y) : y \in [0, 1]\} \subseteq T_{x \leq y}$.

(c) Since $\Pi_y = \{1\}$ it follows that $\Pi \subseteq \{(x, 1) : x \in [0, 1]\} \subseteq T_{x \leq y}$. □

The next result, related with Theorem 6 and with the inclusion relation, characterizes the notion of p-entailment between two conditional events.

Theorem 7. Given two conditional events $A|H, B|K$, with $AH \neq \emptyset$, the following assertions are equivalent:

(a) $A|H \Rightarrow_p B|K$; (b) $A|H \subseteq B|K$, or $K \subseteq B$; (c) $\Pi \subseteq T_{x \leq y}$.

Proof. As $AH \neq \emptyset$, from Theorem 6 the assertions (b) and (c) are equivalent; hence, we only need to show the equivalence between (a) and (b).

((a) \Rightarrow (b)). We will prove that

$$A|H \not\subseteq B|K, AH \neq \emptyset, B^c K \neq \emptyset \implies A|H \not\Rightarrow_p B|K.$$

Assume that $A|H \not\subseteq B|K, B^c K \neq \emptyset$. Then, as in the proof of Theorem 6, we distinguish three cases:

(i) $AHB^c K \neq \emptyset$; (ii) $H^c B^c K \neq \emptyset, AH \neq \emptyset$; (iii) $AHK^c \neq \emptyset, B^c K \neq \emptyset$.

In all three cases the assessment $(1, 0)$ is coherent; thus $A|H \not\Rightarrow_p B|K$.

((b) \Rightarrow (a)). We preliminarily observe that $\{A|H\}$ is p-consistent. Now, if $A|H \subseteq B|K$, then p-entailment of $B|K$ from $A|H$ follows from monotonicity of conditional probability w.r.t. inclusion relation. If $K \subseteq B$, then trivially $A|H$ p-entails $B|K$. \square

Example 1. Given any events A, B , for the conditional events $A \vee B, B|A^c$ it holds that $B|A^c = (A \vee B)|A^c \subseteq A \vee B$. Then, for the assessment $P(A \vee B) = x, P(B|A^c) = y$, the necessary condition of coherence $0 \leq y \leq x \leq 1$ must be satisfied. Of course, $P(A \vee B)$ 'high' does not imply $P(B|A^c)$ 'high'; for instance, it is coherent to assign $P(B|A^c) = 0.2$ and $P(A \vee B) = 0.8$. Then, the inference of the conditional event $B|A^c$ from the disjunction $A \vee B$ may be 'weak'. A probabilistic analysis characterizing the cases in which such an inference is 'strong' has been made in [38].

Remark 4. We observe that, under the hypothesis $A|H \subseteq B|K$, the constituents generated by $\{A|H, B|K\}$ belong to the family

$$\mathcal{H} = \{AHBK, H^c BK, A^c HBK, A^c HK^c, A^c HB^c K, H^c K^c\}.$$

The quasi-conjunction is $C(A|H, B|K) = (AH \vee H^c BK) | (H \vee K)$ and, as shown by Table 1, for any constituent in \mathcal{H} it holds that

$$t(A|H) \leq t(C(A|H, B|K)) \leq t(B|K).$$

Then, we have (see Remark 3)

$$A|H \subseteq B|K \implies A|H \subseteq C(A|H, B|K) \subseteq B|K. \quad (7)$$

As conditional probability is monotonic w.r.t. inclusion relation among conditional events ([45]), it holds that $P(A|H) \leq P[C(A|H, B|K)] \leq P(B|K)$. As shown by Theorem 6, in our coherence-based approach the monotonic property is obtained without assuming that $P(H)$ and $P(K)$ are positive. The next result establishes that $P[C(A|H, B|K)]$ can coherently assume all the values in the interval $[P(A|H), P(B|K)]$. We have

Proposition 1. Let be given any coherent assessment (x, y) on $\{A|H, B|K\}$, with $A|H \subseteq B|K$ and with no further logical relations. Then, the extension $z = P[C(A|H, B|K)]$ is coherent if and only if $l \leq z \leq u$, where

$$l = x = \min(x, y) = T_M(x, y), \quad u = y = \max(x, y) = S_M(x, y).$$

Proof. We recall that, apart from $A|H \subseteq B|K$, there are no further logical relations; thus it holds that $\Pi = T_{x \leq y}$ (i.e. $0 \leq x \leq y \leq 1$). Denoting by $[l, u]$ the interval of coherent extensions of the assessment (x, y) to $C(A|H, B|K)$, by (7) it holds that $[l, u] \subseteq [x, y]$. In order to prove that $[l, u] = [x, y]$ it is enough to verify that both the assessments $\mathcal{P}_l = (x, y, x)$ and $\mathcal{P}_u = (x, y, y)$ are coherent. Given any assessment $\mathcal{P} = (x, y, z)$, with $x \leq y$, we study the coherence of \mathcal{P} by the geometrical approach described in Section 2. The constituents generated by the family and contained in $H \vee K$ are:

$$C_1 = AHBK, C_2 = H^c BK, C_3 = A^c HBK, C_4 = A^c HK^c, C_5 = A^c HB^c K.$$

The corresponding points Q_h 's are

$$Q_1 = (1, 1, 1), Q_2 = (x, 1, 1), Q_3 = (0, 1, 0), Q_4 = (0, y, 0), Q_5 = (0, 0, 0),$$

and, in our case, the coherence of \mathcal{P} simply amounts to the geometrical condition $\mathcal{P} \in \mathcal{I}$, where \mathcal{I} is the convex hull of the points Q_1, Q_2, \dots, Q_5 .

It can be verified that $\mathcal{P}_l = xQ_1 + (y-x)Q_3 + (1-y)Q_5$, so that $\mathcal{P}_l \in \mathcal{I}$; hence $l = x$. Concerning \mathcal{P}_u , we first observe that when $(x, y) = (1, 1)$ we have $\mathcal{P}_u = (1, 1, 1) = Q_1$, so that $\mathcal{P}_u \in \mathcal{I}$; hence $u = y = 1$. Assuming $(x, y) \neq (1, 1)$, it can be verified that $\mathcal{P}_u = \frac{x-xy}{1-x}Q_1 + \frac{y-x}{1-x}Q_2 + (1-y)Q_5$, so that $\mathcal{P}_u \in \mathcal{I}$; hence $u = y$. Therefore, $[l, u] = [x, y]$. \square

We remark that the lower/upper bound above, l, u , may change if we add further logical relations; for instance, if $H = K$, it is $C(A|H, B|H) = A|H$, in which case $l = u = x$. Finally, in agreement with Remark 2, we observe that $T_L(x, y) \leq \min(x, y) \leq \max(x, y) \leq S_0^H(x, y)$. We also recall that $T_M(x, y) = \min(x, y)$ is the largest t-norm and $S_M(x, y) = \max(x, y)$ is the smallest t-conorm ([53]). Figure 2 shows the convex hull \mathcal{I} for given values x, y , with the associated interval $[l, u]$ for $z = P[C(A|H, B|K)]$, when $A|H \subseteq B|K$.

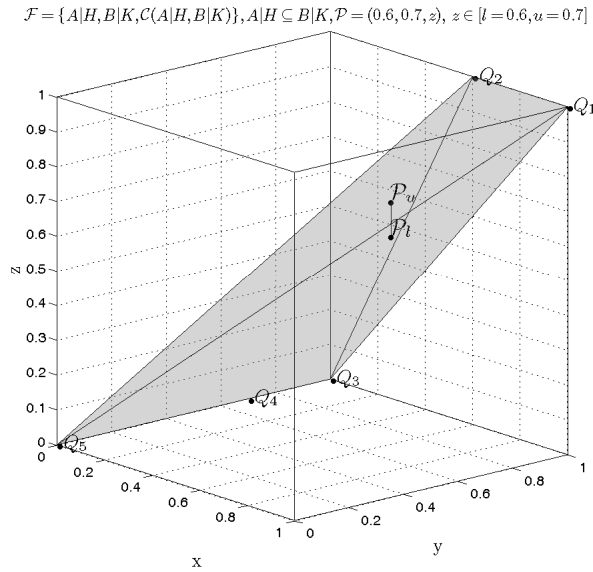


Figure 2: The convex hull \mathcal{I} associated with the pair $(\mathcal{F}, \mathcal{P})$ when $A|H \subseteq B|K$. The interval $[l, u]$ for $z = P[C(A|H, B|K)]$ is the range of the third coordinate z of each $\mathcal{P} \in \overline{\mathcal{P}_l \mathcal{P}_u} = \{(x, y, z) : z \in [T_M(x, y), S_M(x, y)]\}$. The segment $\overline{\mathcal{P}_l \mathcal{P}_u}$ is the intersection between the segment $\{(x, y, z) : z \in [0, 1]\}$ and the convex hull \mathcal{I} . This intersection is empty for $x > y$ because of $\Pi \subseteq T_{x \leq y}$.

3.2. Compound Probability Theorem

We now examine the quasi conjunction of $A|H$ and $B|AH$, with A, B, H logically independent events. As it can be easily verified, we have $C(A|H, B|AH) = AB|H$; moreover, the probability assessment (x, y) on $\{A|H, B|AH\}$ is coherent, for every $(x, y) \in [0, 1]^2$. Hence, by the compound probability theorem, if the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F} = \{A|H, B|AH, AB|H\}$ is coherent, then $z = xy$; i.e., $l = u = x \cdot y = T_P(x, y)$.

In agreement with Remark 2, we observe that $T_L(x, y) \leq xy \leq S_0^H(x, y)$.

We observe that $A|H = AH|H, B|AH = ABH|AH, AB|H = ABH|H$; as $z = xy$, $\{AH|H, ABH|AH\}$ p-entails $ABH|H$ (transitive property). Moreover $AB|H \subseteq B|H$; hence $\{A|H, B|AH\}$ p-entails $B|H$ (cut rule).

3.3. Lower and Upper Bounds for the Quasi Conjunction of n Conditional Events

In this subsection we generalize formula (4). Let be given n conditional events $E_1|H_1, \dots, E_n|H_n$. By the associative property of quasi conjunction, defining $\mathcal{F}_k = \{E_1|H_1, \dots, E_k|H_k\}$, for each $k = 2, \dots, n$ it holds that $C(\mathcal{F}_k) = C(C(\mathcal{F}_{k-1}), E_k|H_k)$. Then, we have

Theorem 8. Given a probability assessment $\mathcal{P}_n = (p_1, p_2, \dots, p_n)$ on $\mathcal{F}_n = \{E_1|H_1, \dots, E_n|H_n\}$, let $[l_k, u_k]$ be the interval of coherent extensions of the assessment $\mathcal{P}_k = (p_1, p_2, \dots, p_k)$ on the quasi conjunction $C(\mathcal{F}_k)$, where $\mathcal{F}_k = \{E_1|H_1, \dots, E_k|H_k\}$. Then, assuming $E_1, H_1, \dots, E_n, H_n$ logically independent, for each $k = 2, \dots, n$, we have

$$l_k = T_L(p_1, p_2, \dots, p_k) = \max(p_1 + p_2 + \dots + p_k - (k-1), 0), \quad (8)$$

$$u_k = S_0^H(p_1, p_2, \dots, p_k) = \begin{cases} 1, & p_i = 1 \text{ for at least one } i, \\ \frac{\sum_{i=1}^k \frac{p_i}{1-p_i}}{\sum_{i=1}^k \frac{p_i}{1-p_i} + 1}, & p_i < 1 \text{ for } i = 1, \dots, k. \end{cases} \quad (9)$$

Proof. Of course, from (4) it is $l_2 = T_L(p_1, p_2)$, $u_2 = S_0^H(p_1, p_2)$. We recall that both T_L and S_0^H are associative. Moreover, as

$$C(\mathcal{F}_3) = C(C(\mathcal{F}_2), E_3|H_3), \quad l_2 \leq P[C(\mathcal{F}_2)] \leq u_2,$$

defining $P[C(\mathcal{F}_2)] = x$ and observing that the quantities $T_L(x, p_3)$, $S_0^H(x, p_3)$ are non-decreasing functions of x , we have

$$\begin{aligned} l_3 &= T_L(l_2, p_3) = T_L(T_L(p_1, p_2), p_3) = T_L(p_1, p_2, p_3), \\ u_3 &= S_0^H(u_2, p_3) = S_0^H(S_0^H(p_1, p_2), p_3) = S_0^H(p_1, p_2, p_3). \end{aligned}$$

Considering any $k > 3$, we proceed by induction. Assuming

$$l_{k-1} = T_L(p_1, p_2, \dots, p_{k-1}), \quad u_{k-1} = S_0^H(p_1, p_2, \dots, p_{k-1}),$$

as $C(\mathcal{F}_k) = C(C(\mathcal{F}_{k-1}), E_k|H_k)$ and $l_{k-1} \leq P[C(\mathcal{F}_{k-1})] \leq u_{k-1}$, defining $P[C(\mathcal{F}_{k-1})] = x$ and observing that the quantities $T_L(x, p_k)$ and $S_0^H(x, p_k)$ are non-decreasing functions of x , we have

$$\begin{aligned} l_k &= T_L(l_{k-1}, p_k) = T_L(p_1, p_2, \dots, p_k), \\ u_k &= S_0^H(u_{k-1}, p_k) = S_0^H(p_1, p_2, \dots, p_k). \end{aligned}$$

The explicit values of l_k and u_k in (8) and (9) follow by Appendix B and Appendix C. \square

Notice that $(p_1, p_2, \dots, p_n) = (1, 1, \dots, 1)$ implies $T_L(p_1, p_2, \dots, p_n) = 1$. Then, from Theorem 8, we obtain the following *Quasi And* rule (see also [41, Theorem 4]).

Corollary 1. Given a p-consistent family of conditional events \mathcal{F}_n , we have

$$(\text{Quasi And}) \quad \mathcal{F}_n \Rightarrow_p C(\mathcal{F}_n). \quad (10)$$

We observe that, from (1), we obtain ([42, Theorem 5])

$$\mathcal{F}_n \Rightarrow_p C(S), \quad \forall S \subseteq \mathcal{F}_n. \quad (11)$$

Of course, (11) still holds when there are logical dependencies because in this case the lower bound for quasi conjunction does not decrease, as observed in Remark 2. In the next example we illustrate the key role of quasi conjunction when we study p-entailment. This example has been already examined in [35], by using the inference rules of System P in the setting of coherence.

Example 2 (Linda's example). We start with a given p-consistent family of conditional events \mathcal{F} ; then, we use the quasi conjunction to check the p-entailment of some further conditional events from \mathcal{F} . The family \mathcal{F} concerns various attributes for a given party (*the party is great*, *the party is noisy*, *Linda and Steve are present*, and so on). We introduce the following events:

$$\begin{aligned} L &= \text{"Linda goes to the party"}; & S &= \text{"Steve goes to the party"}; \\ G &= \text{"the party is great"}; & N &= \text{"the party is noisy"}, \end{aligned}$$

which are assumed to be logically independent. Then, we consider the family $\mathcal{F} = \{G|L, S|L, N^c|LS, L|S, G^c|N^c\}$ and the family of further conditional events

$$\mathcal{K} = \{N^c|L, L^c|\Omega, GN^c|LS, N^c|S, N^c|(L \vee S)\}.$$

It can be verified that the assessment $(1, 1, 1, 1, 1)$ on \mathcal{F} is coherent, i.e. the family \mathcal{F} is p-consistent. By exploiting quasi conjunction, we can verify that \mathcal{F} p-entails \mathcal{K} ; that is \mathcal{F} p-entails each conditional event in \mathcal{K} . Indeed:

- (a) concerning $N^c|L$, for the subset $\mathcal{S} = \{S|L, N^c|LS\}$ we have $C(\mathcal{S}) = N^cS|L \subseteq N^c|L$; thus: $\mathcal{F} \Rightarrow_p C(\mathcal{S}) \Rightarrow_p N^c|L$;
- (b) concerning $L^c|\Omega$, for the subset $\mathcal{S} = \{G|L, S|L, N^c|LS, G^c|N^c\}$ we have $C(\mathcal{S}) = G^cL^cN^c|(L \vee N^c) \subseteq L^c|\Omega$; thus: $\mathcal{F} \Rightarrow_p C(\mathcal{S}) \Rightarrow_p L^c|\Omega$;
- (c) concerning $GN^c|LS$, for the subset $\mathcal{S} = \{G|L, S|L, N^c|LS\}$ we have $C(\mathcal{S}) = GN^cS|L \subseteq GN^c|LS$; thus: $\mathcal{F} \Rightarrow_p C(\mathcal{S}) \Rightarrow_p GN^c|LS$;
- (d) concerning $N^c|S$, for the subset $\mathcal{S} = \{N^c|LS, L|S\}$ we have $C(\mathcal{S}) = LN^c|S \subseteq N^c|S$; thus: $\mathcal{F} \Rightarrow_p C(\mathcal{S}) \Rightarrow_p N^c|S$;
- (e) concerning $N^c|(L \vee S)$, for the subset $\mathcal{S} = \{S|L, N^c|LS, L|S\}$ we have $C(\mathcal{S}) = LN^cS|(L \vee S) \subseteq N^c|(L \vee S)$; thus: $\mathcal{F} \Rightarrow_p C(\mathcal{S}) \Rightarrow_p N^c|(L \vee S)$.

We point out that the p-entailment of \mathcal{K} from \mathcal{F} can be also verified by applying Algorithm 2 in [42]. We also observe that, using the basic events L, S, G, N , we can define conditional events which are not p-entailed from \mathcal{F} . For instance, concerning $G|N$, associated with the conditional “if the party is noisy, then the party is great”, it can be proved that \mathcal{F} does not p-entail $G|N$. Indeed, there is no subset $\mathcal{S} \subseteq \mathcal{F}$, with $\mathcal{S} \neq \emptyset$, such that $C(\mathcal{S}) \Rightarrow_p G|N$ (see [42, Theorem 6]).

3.4. The Case $E_1|H_1 \subseteq E_2|H_2 \subseteq \dots \subseteq E_n|H_n$

In this subsection we give a result on quasi conjunctions when $E_i|H_i \subseteq E_{i+1}|H_{i+1}, i = 1, \dots, n-1$. We have

Theorem 9. Given a family $\mathcal{F}_n = \{E_1|H_1, \dots, E_n|H_n\}$ of conditional events such that $E_1|H_1 \subseteq E_2|H_2 \subseteq \dots \subseteq E_n|H_n$, and a coherent probability assessment $\mathcal{P}_n = (p_1, p_2, \dots, p_n)$ on \mathcal{F}_n , let $C(\mathcal{F}_k)$ be the quasi conjunction of $\mathcal{F}_k = \{E_i|H_i, i = 1, \dots, k\}$, $k = 2, \dots, n$. Moreover, let $[l_k, u_k]$ be the interval of coherent extensions on $C(\mathcal{F}_k)$ of the assessment (p_1, p_2, \dots, p_k) on \mathcal{F}_k . We have: (i) $E_1|H_1 \subseteq C(\mathcal{F}_2) \subseteq \dots \subseteq C(\mathcal{F}_n) \subseteq E_n|H_n$; (ii) by assuming no further logical relations, any probability assessment (z_2, \dots, z_k) on $\{C(\mathcal{F}_2), \dots, C(\mathcal{F}_k)\}$ is a coherent extension of the assessment (p_1, p_2, \dots, p_k) on \mathcal{F}_k if and only if $p_1 \leq z_2 \leq \dots \leq z_k \leq p_k$, $k = 2, \dots, n$; moreover

$$l_k = \min(p_1, \dots, p_k) = p_1, \quad u_k = \max(p_1, \dots, p_k) = p_k, \quad k = 2, \dots, n.$$

Proof. (i) By iteratively applying (7) and by the associative property of quasi conjunction, we have $C(\mathcal{F}_{k-1}) \subseteq C(\mathcal{F}_k) \subseteq E_k|H_k$, $k = 2, \dots, n$;

(ii) by exploiting the logical relations in point (i), the assertions immediately follow by applying a reasoning similar to that in Remark 4. \square

3.5. Generalized Compound Probability Theorem

In this subsection we generalize the result obtained in Subsection 3.2. Given the family $\mathcal{F} = \{A_1|H, A_2|A_1H, \dots, A_n|A_1 \cdots A_{n-1}H\}$, by iteratively exploiting the associative property, we have

$$\begin{aligned} C(\mathcal{F}) &= C(C(A_1|H, A_2|A_1H), A_3|A_1A_2H, \dots, A_n|A_1 \cdots A_{n-1}H) = \\ &= C(A_1A_2|H, A_3|A_1A_2H, \dots, A_n|A_1 \cdots A_{n-1}H) = \dots = A_1A_2 \cdots A_n|H; \end{aligned}$$

thus, by the compound probability theorem, if the assessment $\mathcal{P} = (p_1, \dots, p_n, z)$ on $\mathcal{F} \cup \{C(\mathcal{F})\}$ is coherent, then

$$z = l = u = p_1p_2 \cdots p_n = T_P(p_1, p_2, \dots, p_n).$$

4. Further Aspects on Quasi Conjunction: from Bounds on Conclusions to Bounds on Premises in Quasi And rule

In this section, we study the propagation of probability bounds on the conclusion of the Quasi And rule to its premises. We start with the case of two premises $A|H$ and $B|K$, by examining probabilistic aspects on the lower and upper bounds, l and u , for the probability of the conclusion $C(A|H, B|K)$. More precisely, given any number $\gamma \in [0, 1]$, we find:

- (i) the set \mathcal{L}_γ of the coherent assessments (x, y) on $\{A|H, B|K\}$ such that, for each $(x, y) \in \mathcal{L}_\gamma$, one has $l \geq \gamma$;
- (ii) the set \mathcal{U}_γ of the coherent assessments (x, y) on $\{A|H, B|K\}$ such that, for each $(x, y) \in \mathcal{U}_\gamma$, one has $u \leq \gamma$.

Case (i). Of course, $\mathcal{L}_0 = [0, 1]^2$; hence we can assume $\gamma > 0$. It must be $l = \max\{x + y - 1, 0\} \geq \gamma$, i.e., $x + y \geq 1 + \gamma$ (as $\gamma > 0$); hence \mathcal{L}_γ coincides with the triangle having the vertices $(1, 1)$, $(1, \gamma)$, $(\gamma, 1)$; that is

$$\mathcal{L}_\gamma = \{(x, y) : \gamma \leq x \leq 1, 1 + \gamma - x \leq y \leq 1\}.$$

Notice that $\mathcal{L}_1 = \{(1, 1)\}$; moreover, for $\gamma \in (0, 1)$, $(\gamma, \gamma) \notin \mathcal{L}_\gamma$.

Case (ii). Of course, $\mathcal{U}_1 = [0, 1]^2$; hence we can assume $\gamma < 1$. We recall that $u = S_0^H(x, y)$, then in order the inequality $S_0^H(x, y) \leq \gamma$ be satisfied, it must be $x < 1$ and $y < 1$. Thus, $u \leq \gamma$ if and only if $\frac{x+y-2xy}{1-xy} \leq \gamma$. Given any $x < 1, y < 1$, we have

$$u - x = \frac{y(1-x)^2}{1-xy} \geq 0, \quad u - y = \frac{x(1-y)^2}{1-xy} \geq 0; \quad (12)$$

then, from $u \leq \gamma$ it follows $x \leq \gamma, y \leq \gamma$; hence $\mathcal{U}_\gamma \subseteq [0, \gamma]^2$. Then, taking into account that $x \leq \gamma$ and hence

$$1 - (2 - \gamma)x = 1 - 2x + \gamma x \geq 1 - 2x + x^2 = (1 - x)^2 > 0,$$

we have

$$\frac{x + y - 2xy}{1 - xy} \leq \gamma \iff y \leq \frac{\gamma - x}{1 - (2 - \gamma)x}; \quad (13)$$

therefore

$$\mathcal{U}_\gamma = \left\{ (x, y) : 0 \leq x \leq \gamma, y \leq \frac{\gamma - x}{1 - (2 - \gamma)x} \right\}.$$

Notice that $\mathcal{U}_0 = \{(0, 0)\}$; moreover, for $x = y = \gamma \in (0, 1)$, it is $u = \frac{2\gamma}{1+\gamma} > \gamma$; hence, for $\gamma \in (0, 1)$, \mathcal{U}_γ is a strict subset of $[0, \gamma]^2$.

Of course, for every $(x, y) \notin \mathcal{L}_\gamma \cup \mathcal{U}_\gamma$, it is $l < \gamma < u$. Figure 3 displays the sets $\mathcal{L}_\gamma, \mathcal{U}_\gamma$ when $\gamma = 0.6$.

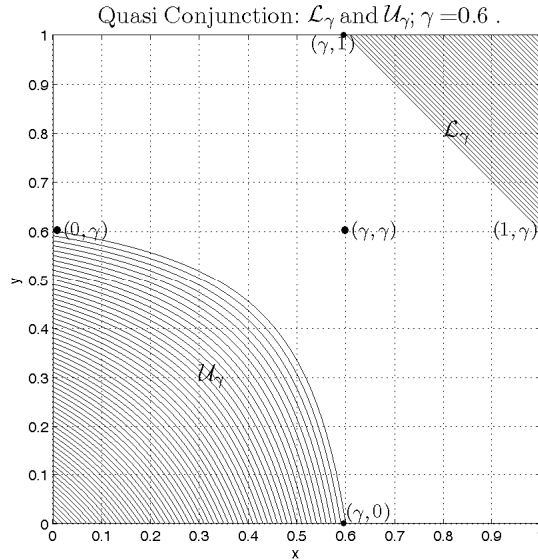


Figure 3: The sets $\mathcal{L}_\gamma, \mathcal{U}_\gamma$.

In the next result we determine in general the sets $\mathcal{L}_\gamma, \mathcal{U}_\gamma$.

Theorem 10. Given a coherent assessment (p_1, p_2, \dots, p_n) on the family $\{E_1|H_1, \dots, E_n|H_n\}$, where $E_1, H_1, \dots, E_n, H_n$ are logically independent, we have

$$\mathcal{L}_\gamma = \{(p_1, \dots, p_n) \in [0, 1]^n : p_1 + \dots + p_n \geq \gamma + n - 1\}, \quad \gamma > 0, \quad (14)$$

$$\mathcal{U}_\gamma = \{(p_1, \dots, p_n) \in [0, 1]^n : 0 \leq p_1 \leq \gamma, p_{k+1} \leq r_k, k = 1, \dots, n-1\}, \quad \gamma < 1,$$

where $r_k = \frac{\gamma - u_k}{1 - (2 - \gamma)u_k}$, $u_k = S_0^H(p_1, \dots, p_k)$, with $\mathcal{L}_0 = \mathcal{U}_1 = [0, 1]^n$.

Proof. Of course, $\mathcal{L}_0 = [0, 1]^n$, so that we can assume $\gamma > 0$. It must be $l_n = \max(p_1 + \dots + p_n - (n - 1), 0) \geq \gamma$, that is, as $\gamma > 0$, $p_1 + \dots + p_n \geq \gamma + n - 1$. Hence: $\mathcal{L}_\gamma = \{(p_1, \dots, p_n) \in [0, 1]^n : p_1 + \dots + p_n \geq \gamma + n - 1\}$.

We observe that \mathcal{L}_γ is a convex polyhedron with vertices the points

$$\begin{aligned} V_1 &= (\gamma, 1, \dots, 1), \quad V_2 = (1, \gamma, 1, \dots, 1), \quad \dots, \\ V_n &= (1, \dots, 1, \gamma), \quad V_{n+1} = (1, 1, \dots, 1). \end{aligned}$$

Moreover, the convex hull of the vertices V_1, \dots, V_n is the subset of the points (p_1, \dots, p_n) of \mathcal{L}_γ such that $l_n = \gamma$, that is such that $p_1 + \dots + p_n = \gamma + n - 1$.

Now, let us determine the set \mathcal{U}_γ . Of course, $\mathcal{U}_1 = [0, 1]^n$, so that we can assume $\gamma < 1$. We recall that u_2, \dots, u_n are the upper bounds on $C(\mathcal{F}_2), \dots, C(\mathcal{F}_n)$ associated with (p_1, \dots, p_n) . Then, from the relations

$$C(\mathcal{F}_{k+1}) = C(C(\mathcal{F}_k), E_{k+1}|H_{k+1}), \quad k = 2, \dots, n - 1,$$

by applying (12) with $x = u_k, y = p_{k+1}$, we have that in order the inequality $u_{k+1} \leq \gamma$ be satisfied, it must be $u_k \leq \gamma, p_{k+1} \leq \gamma, k = 2, \dots, n - 1$. Therefore

$$u_n \leq \gamma \implies p_1 \leq \gamma, \dots, p_n \leq \gamma, u_2 \leq \gamma, \dots, u_{n-1} \leq \gamma,$$

so that $\mathcal{U}_\gamma \subseteq [0, \gamma]^n$. By iteratively applying (13), we obtain

$$\begin{aligned} 0 \leq p_1 \leq \gamma, \quad p_2 &\leq \frac{\gamma - p_1}{1 - (2 - \gamma)p_1} \implies u_2 \leq \gamma, \\ 0 \leq u_2 \leq \gamma, \quad p_3 &\leq \frac{\gamma - u_2}{1 - (2 - \gamma)u_2} \implies u_3 \leq \gamma, \\ &\vdots \\ 0 \leq u_{n-1} \leq \gamma, \quad p_n &\leq \frac{\gamma - u_{n-1}}{1 - (2 - \gamma)u_{n-1}} \implies u_n \leq \gamma. \end{aligned}$$

Therefore, observing that $u_1 = p_1$,

$$\mathcal{U}_\gamma = \{(p_1, \dots, p_n) \in [0, 1]^n : 0 \leq p_1 \leq \gamma, \quad p_{k+1} \leq \frac{\gamma - u_k}{1 - (2 - \gamma)u_k}, \quad k = 1, \dots, n - 1\}.$$

We observe that $\mathcal{U}_0 = \{(0, \dots, 0)\}$; moreover, for $p_1 = \dots = p_n = \gamma \in (0, 1)$, we obtain (by induction)

$$u_2 = \frac{2\gamma}{1 + \gamma} > \gamma, \quad u_3 = \frac{3\gamma}{1 + 2\gamma} > \gamma, \quad \dots, \quad u_n = \frac{n\gamma}{1 + (n - 1)\gamma} > \gamma;$$

hence, for $\gamma \in (0, 1)$, \mathcal{U}_γ is a strict subset of $[0, \gamma]^n$.

Of course, for every $(p_1, \dots, p_n) \notin \mathcal{L}_\gamma \cup \mathcal{U}_\gamma$, it is $l_n < \gamma < u_n$. As an example, for $p_1 = \dots = p_n = \gamma \in (0, 1)$, one has

$$l_n = \max(n\gamma - (n - 1), 0) < \gamma < u_n = \frac{n\gamma}{1 + (n - 1)\gamma}.$$

□

5. Lower and Upper Bounds for Quasi Disjunction

We recall below the notion of quasi disjunction of conditional events as defined in [1].

Definition 8. Given any events A, H, B, K , with $H \neq \emptyset, K \neq \emptyset$, the quasi disjunction of the conditional events $A|H$ and $B|K$ is the conditional event $\mathcal{D}(A|H, B|K) = (AH \vee BK)|(H \vee K)$.

The constituents generated by the family $\{A|H, B|K, \mathcal{D}(A|H, B|K)\}$ and the corresponding points Q_h 's are given in columns 2 and 8 of Table 1. In general, given a family of n conditional events $\mathcal{F}_n = \{E_i|H_i, i = 1, \dots, n\}$, it is $\mathcal{D}(\mathcal{F}_n) = \mathcal{D}(E_1|H_1, \dots, E_n|H_n) = (\bigvee_{i=1}^n E_i H_i) | (\bigvee_{i=1}^n H_i)$. Quasi disjunction is associative; that is, for every subset $J \subset \{1, \dots, n\}$, we have $\mathcal{D}(\mathcal{F}_n) = \mathcal{D}(\mathcal{F}_J \cup \mathcal{F}_{\Gamma}) = \mathcal{D}[\mathcal{D}(\mathcal{F}_J), \mathcal{D}(\mathcal{F}_{\Gamma})]$, where $\Gamma = \{1, \dots, n\} \setminus J$.

Remark 5. We recall that the quasi conjunction of $A|H$ and $B|K$ can also be written as $C(A|H, B|K) = (A \vee H^c) \wedge (B \vee K^c) | (H \vee K)$; then, based on the usual negation operation $(E|H)^c = E^c|H$, it holds that

$$\begin{aligned} [C(A^c|H, B^c|K)]^c &= [(A^c \vee H^c) \wedge (B^c \vee K^c) | (H \vee K)]^c = \\ &= (AH \vee BK) | (H \vee K) = \mathcal{D}(A|H, B|K), \end{aligned} \quad (15)$$

which represents the De Morgan duality between quasi conjunction and quasi disjunction. We also have $\mathcal{D}(A|H, B|K) \vee C(A^c|H, B^c|K) = \Omega | (H \vee K)$ and $\mathcal{D}(A|H, B|K) \wedge C(A^c|H, B^c|K) = \emptyset | (H \vee K)$. From (15) it follows

$$P[\mathcal{D}(A|H, B|K)] = 1 - P[C(A^c|H, B^c|K)], \quad (16)$$

which will be exploited in the next result.

Proposition 2. Assuming A, H, B, K logically independent, we have:

- (i) the probability assessment (x, y) on $\{A|H, B|K\}$ is coherent for every $(x, y) \in [0, 1]^2$;
- (ii) given a coherent assessment (x, y) on $\{A|H, B|K\}$, the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F} = \{A|H, B|K, \mathcal{D}(A|H, B|K)\}$, with $z = P[\mathcal{D}(A|H, B|K)]$, is a coherent extension of (x, y) if and only if $z \in [l, u]$, where

$$l = T_0^H(x, y) = \begin{cases} \frac{xy}{x+y-xy}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0), \end{cases} \quad u = S_L(x, y) = \min(x + y, 1).$$

Proof. We observe that, by (4), the extension $\gamma = P[C(A^c|H, B^c|K)]$ of the assessment $P(A|H) = x, P(B|K) = y$ is coherent if and only if $\gamma' \leq \gamma \leq \gamma''$, where $\gamma' = T_L(1 - x, 1 - y)$, $\gamma'' = S_0^H(1 - x, 1 - y)$. Then, based on (16) and on the results given in Appendix A, it follows that

$$l = 1 - S_0^H(1 - x, 1 - y) = T_0^H(x, y), \quad u = 1 - T_L(1 - x, 1 - y) = S_L(x, y).$$

□

In Figure 4 is shown the convex hull \mathcal{I} for given values x, y , with the associated interval $[l, u]$ of coherent extensions $z = P[\mathcal{D}(A|H, B|K)]$. As for quasi conjunction, the convex hull \mathcal{I} does not depend on z . In the next subsections we examine some particular cases.

5.1. The Dual of Compound Probability Theorem

Given any logically independent events A, B, H , with $A^c H \neq \emptyset$, the assessment (x, y) on $\{A|H, B|A^c H\}$ is coherent, for every $(x, y) \in [0, 1]^2$. We have $\mathcal{D}(A|H, B|A^c H) = (A \vee B)|H$ and, defining $z = P(A \vee B|H)$, by (16) and by the results in Subsection 3.2, we have

$$\begin{aligned} z &= P(\mathcal{D}(A|H, B|A^c H)) = 1 - P(C(A^c|H, B^c|A^c H)) = \\ &= 1 - T_P(1 - x, 1 - y) = x + y - xy = S_P(x, y); \end{aligned}$$

that is z is equal to the *probabilistic sum* of x, y .

5.2. The case $A|H \subseteq B|K$

From $A|H \subseteq B|K$ we have $\mathcal{D}(A|H, B|K) = (BK) | (H \vee K)$. Then, as shown by Table 1 and by Remark 4), it holds that

$$t(A|H) \leq t(C(A|H, B|K)) \leq t(\mathcal{D}(A|H, B|K)) \leq t(B|K).$$

Then: $A|H \subseteq B|K$ implies $A|H \subseteq C(A|H, B|K) \subseteq \mathcal{D}(A|H, B|K) \subseteq B|K$. We recall that, by Remark 3, $A|H \subseteq B|K$ amounts to $B^c|K \subseteq A^c|H$; then, given the assessment $P(A|H) = x, P(B|K) = y$, where $x \leq y$, by applying Proposition 1 to the family $\{B^c|K, A^c|H\}$, the extension $\gamma = P[C(B^c|K, A^c|H)]$ of (x, y) is coherent if and only if $\gamma' \leq \gamma \leq \gamma''$, where $\gamma' = 1 - y, \gamma'' = 1 - x$. Then, by (16), the extension $z = P[\mathcal{D}(A|H, B|K)]$ of (x, y) is coherent if and only if $l \leq z \leq u$, where $l = x = \min(x, y), u = y = \max(x, y)$.

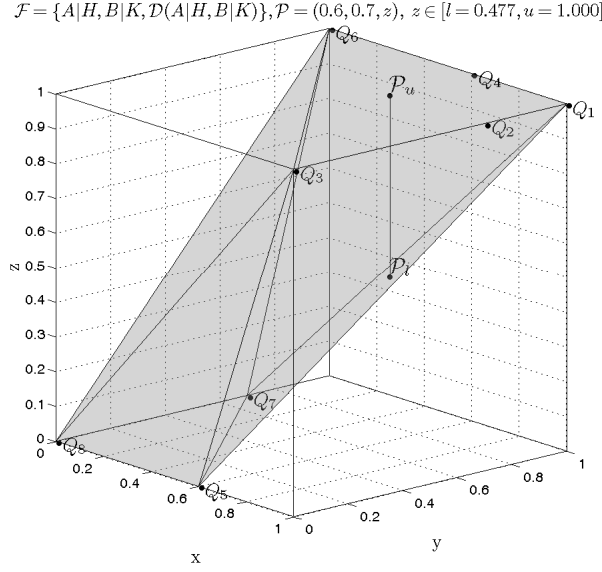


Figure 4: The convex hull \mathcal{I} associated with the pair $(\mathcal{F}, \mathcal{P})$ in case of quasi disjunction without logical relations. The interval $[l, u]$ for $z = P[\mathcal{D}(A|H, B|K)]$ is the range of the third coordinate z of each $\mathcal{P} \in \overline{\mathcal{P}_l \mathcal{P}_u} = \{(x, y, z) : z \in [T_0^H(x, y), S_L(x, y)]\}$. The segment $\overline{\mathcal{P}_l \mathcal{P}_u}$ is the intersection between the segment $\{(x, y, z) : z \in [0, 1]\}$ and the convex hull \mathcal{I} .

5.3. Quasi Conjunction, Quasi Disjunction and Or Rule.

We recall that in Or rule with premises $H \sim A$ and $K \sim A$ the conclusion is $H \vee K \sim A$. Moreover, for the conditional events $A|H$ and $A|K$ associated with the premises, we have

$$C(A|H, A|K) = \mathcal{D}(A|H, A|K) = A|(H \vee K),$$

which is the conditional event associated with the conclusion $H \vee K \sim A$ of Or rule. In [35] it has been proved that, under logical independence of A, H, K , the assessment $z = P(A|(H \vee K))$ is a coherent extension of the assessment (x, y) on $\{A|H, A|K\}$ if and only if $z \in [l, u]$, with

$$l = T_0^H(x, y), \quad u = S_0^H(x, y). \quad (17)$$

The convex hull \mathcal{I} for given values x, y and the associated interval $[l, u]$ for $z = P[\mathcal{D}(A|H, A|K)]$ are shown in Figure 5.

5.4. Lower and Upper Bounds for the Quasi Disjunction of n Conditional Events

Given the family $\mathcal{F}_n = \{E_1|H_1, \dots, E_n|H_n\}$, let us consider the quasi disjunction $\mathcal{D}(\mathcal{F}_n)$ of the conditional events in \mathcal{F}_n . By the associative property of quasi disjunction, defining $\mathcal{F}_k = \{E_1|H_1, \dots, E_k|H_k\}$, for each $k = 2, \dots, n$ it holds that $\mathcal{D}(\mathcal{F}_k) = \mathcal{D}(\mathcal{D}(\mathcal{F}_{k-1}), E_k|H_k)$. Then, denoting by T_0^H the Hamacher t-norm with parameter $\lambda = 0$ and by S_L the Lukasiewicz t-conorm (see Appendix B and Appendix C), we have

Theorem 11. Given a probability assessment $\mathcal{P}_n = (p_1, p_2, \dots, p_n)$ on $\mathcal{F}_n = \{E_1|H_1, \dots, E_n|H_n\}$, let $[l_k, u_k]$ be the interval of coherent extensions of the assessment $\mathcal{P}_k = (p_1, p_2, \dots, p_k)$ on the quasi disjunction $\mathcal{D}(\mathcal{F}_k)$, where $\mathcal{F}_k = \{E_1|H_1, \dots, E_k|H_k\}$. Then, assuming $E_1, H_1, \dots, E_n, H_n$ logically independent, for each $k = 2, \dots, n$, we have

$$l_k = T_0^H(p_1, p_2, \dots, p_k), \quad u_k = S_L(p_1, p_2, \dots, p_k).$$

Proof. Of course, from Proposition 2 it is $l_2 = T_0^H(p_1, p_2)$ and $u_2 = S_L(p_1, p_2)$. The rest of the proof is similar to that one in Theorem 8. \square

$$\mathcal{F} = \{A|H, A|K, A|(H \vee K)\}, \mathcal{P} = (0.6, 0.7, z), z \in [l = 0.477, u = 0.793]$$

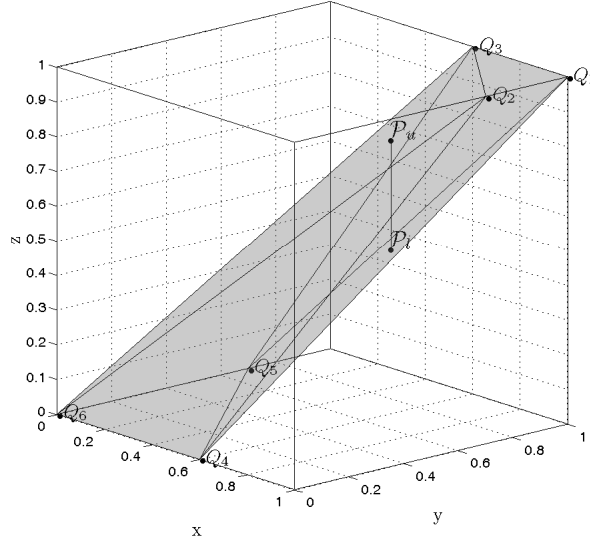


Figure 5: Convex hull I associated with the pair $(\mathcal{F}, \mathcal{P})$ for the Or rule. The interval $[l, u]$ for $z = P[\mathcal{D}(A|H, A|K)] = P[C(A|H, A|K)]$ is the range of the third coordinate z of each $\mathcal{P} \in \overline{\mathcal{P}_l \mathcal{P}_u} = \{(x, y, z) : z \in [T_0^H(x, y), S_0^H(x, y)]\}$. The segment $\overline{\mathcal{P}_l \mathcal{P}_u}$ is the intersection between the segment $\{(x, y, z) : z \in [0, 1]\}$ and the convex hull I .

Remark 6. Given any conditional events $A|H$ and $B|K$, as shown in Table 1, it holds that $t(C(A|H, B|K)) \leq t(\mathcal{D}(A|H, B|K))$, which amounts to $C(A|H, B|K) \subseteq \mathcal{D}(A|H, B|K)$. In general, given a finite family of conditional events \mathcal{F}_n , we have $t(C(\mathcal{F}_n)) \leq t(\mathcal{D}(\mathcal{F}_n))$, that is $C(\mathcal{F}_n) \subseteq \mathcal{D}(\mathcal{F}_n)$, so that $P[C(\mathcal{F}_n)] \leq P[\mathcal{D}(\mathcal{F}_n)]$. Thus, if the family \mathcal{F}_n is p-consistent, then $\mathcal{F}_n \Rightarrow_p C(\mathcal{F}_n) \Rightarrow_p \mathcal{D}(\mathcal{F}_n)$ and we obtain the following *Quasi Or* rule

$$(\text{Quasi Or}) \quad \mathcal{F}_n \Rightarrow_p \mathcal{D}(\mathcal{F}_n). \quad (18)$$

We observe that Quasi Or rule also follows directly from Theorem 11.

5.5. General Or Rule

Let us consider the general Or rule (see [37]), where the premises are the conditional events $E|H_1, \dots, E|H_n$ and the conclusion is the conditional event $E|(H_1 \vee H_2 \vee \dots \vee H_n)$. By the associative property of quasi disjunction, defining $\mathcal{F}_k = \{E|H_1, \dots, E|H_k\}$, for each $k = 2, \dots, n$ it holds that

$$\mathcal{D}(\mathcal{F}_k) = \mathcal{D}(\mathcal{D}(\mathcal{F}_{k-1}), E|H_k) = E|(H_1 \vee H_2 \vee \dots \vee H_k).$$

We also observe that $\mathcal{D}(\mathcal{F}_k) = C(\mathcal{F}_k)$. Then, by exploiting the notions of t-norm, t-conorm, quasi disjunction and quasi conjunction, Theorem 9 in [37] can be written as

Theorem 12. Given a probability assessment $\mathcal{P}_n = (p_1, p_2, \dots, p_n)$ on $\mathcal{F}_n = \{E|H_1, E|H_2, \dots, E|H_n\}$, let $[l_k, u_k]$ be the interval of coherent extensions of the assessment $\mathcal{P}_k = (p_1, p_2, \dots, p_k)$ on the quasi disjunction $\mathcal{D}(\mathcal{F}_k)$, where $\mathcal{F}_k = \{E|H_1, \dots, E|H_k\}$. Then, assuming E, H_1, \dots, H_n logically independent, for each $k = 2, \dots, n$, we have

$$l_k = T_0^H(p_1, p_2, \dots, p_k), \quad u_k = S_0^H(p_1, p_2, \dots, p_k).$$

Proof. Of course, from (17) it is $l_2 = T_0^H(p_1, p_2)$ and $u_2 = S_0^H(p_1, p_2)$. The rest of the proof is similar to that one in Theorem 8. \square

In [37, Theorem 9]), by implicitly assuming $(p_1, \dots, p_k) \in (0, 1)^k$, it has been proved by a direct probabilistic analysis that

$$l_k = \frac{1}{1 + \sum_{i=1}^k \frac{1-p_i}{p_i}}, \quad u_k = \frac{\sum_{i=1}^k \frac{p_i}{1-p_i}}{1 + \sum_{i=1}^k \frac{p_i}{1-p_i}}.$$

By adopting the conventions $\frac{1}{\infty} = 0$, $\frac{1}{0} = \infty$, $\frac{\infty}{\infty} = 1$, the previous formulas hold in general for every $(p_1, \dots, p_k) \in [0, 1]^k$. In Appendix C the previous expressions for the Hamacher t-norm and t-conorm have been derived by using the notion of additive generator.

Example 3. An application of Or rule is obtained by imagining a medical scenario with a disease E and n symptoms H_1, \dots, H_n , with $P(E|H_i) = p_i$, $i = 1, \dots, n$, and $P(E|(H_1 \vee \dots \vee H_n)) \in [l_n, u_n]$. If, for instance, $p_1 = \dots = p_n = 1 - \varepsilon$, from Theorem 12 it follows $l_n = T_0^H(1 - \varepsilon, \dots, 1 - \varepsilon) = \frac{1-\varepsilon}{1+(n-1)\varepsilon}$ and $u_n = S_0^H(1 - \varepsilon, \dots, 1 - \varepsilon) = \frac{n(1-\varepsilon)}{\varepsilon+n(1-\varepsilon)}$. Then: (i) for $\varepsilon \rightarrow 0$, we have $l_n \rightarrow 1$ and $u_n \rightarrow 1$; (ii) for $n \rightarrow +\infty$ we have $l_n \rightarrow 0$ and $u_n \rightarrow 1$. As we can see, in the second case the interval $[l_n, u_n]$ gets wider and wider as the number of premises increases. An interesting related phenomenon where additional information leads to less informative conclusion is the pseudodiagnosticity task, studied in the psychology of uncertain reasoning ([51, 66]).

6. Further Aspects on Quasi Disjunction: from Bounds on Conclusions to Bounds on Premises in Quasi Or rule

In this section, we study the propagation of probability bounds on the conclusion of the Quasi Or rule to its premises. We start with the case of two premises $A|H$ and $B|K$, by examining probabilistic aspects on the lower and upper bounds, l and u , for the probability of the conclusion $\mathcal{D}(A|H, B|K)$. More precisely, given any number $\gamma \in [0, 1]$, we find:

- (i) the set \mathbf{L}_γ of the coherent assessments (x, y) on $\{A|H, B|K\}$ such that, for each $(x, y) \in \mathbf{L}_\gamma$, one has $l \geq \gamma$;
- (ii) the set \mathbf{U}_γ of the coherent assessments (x, y) on $\{A|H, B|K\}$ such that, for each $(x, y) \in \mathbf{U}_\gamma$, one has $u \leq \gamma$.

Case (i). Let be given $\gamma \in [0, 1]$. We denote by \mathbf{L}_γ the set of coherent assessments (x, y) on $\{A|H, B|K\}$ which imply $z \geq \gamma$. Of course, $\mathbf{L}_0 = [0, 1]^2$; hence we can assume $\gamma > 0$. We recall that $l = T_0^H(x, y)$, then in order the inequality $T_0^H(x, y) \geq \gamma$ be satisfied, it must be $x > 0$ and $y > 0$. Thus, $l \geq \gamma$ if and only if $\frac{xy}{x+y-xy} \geq \gamma$. We have

$$x - l = \frac{x^2(1-y)}{x+y-xy} \geq 0, \quad y - l = \frac{y^2(1-x)}{x+y-xy} \geq 0; \quad (19)$$

then, from $l \geq \gamma$ it follows $x \geq \gamma$, $y \geq \gamma$; thus $\mathbf{L}_\gamma \subseteq [\gamma, 1]^2$. Then, taking into account that $x \geq \gamma$ and hence $x(1+\gamma) - \gamma > 0$, we have

$$\frac{xy}{x+y-xy} \geq \gamma \iff y \geq \frac{\gamma x}{x(1+\gamma) - \gamma}; \quad (20)$$

therefore

$$\mathbf{L}_\gamma = \left\{ (x, y) : \gamma \leq x \leq 1, \quad y \geq \frac{\gamma x}{x - \gamma + \gamma x} \right\}.$$

Notice that $\mathbf{L}_1 = \{(1, 1)\}$; for $x = y = \gamma \in (0, 1)$, it is $l = \frac{\gamma}{2-\gamma} < \gamma$; hence, for $\gamma \in (0, 1)$, \mathbf{L}_γ is a strict subset of $[\gamma, 1]^2$.

Case (ii). Of course, $\mathbf{U}_1 = [0, 1]^2$; hence we can assume $\gamma < 1$. It must be $u = \min\{x+y, 1\} \leq \gamma$, i.e., $x+y \leq \gamma$ (as $\gamma < 1$); hence \mathbf{U}_γ coincides with the triangle having the vertices $(0, 0)$, $(0, \gamma)$, $(\gamma, 0)$; that is

$$\mathbf{U}_\gamma = \{(x, y) : 0 \leq x \leq \gamma, \quad 0 \leq y \leq \gamma - x\}.$$

Notice that $\mathbf{U}_0 = \{(0, 0)\}$; moreover, for $\gamma \in (0, 1)$, $(\gamma, \gamma) \notin \mathbf{U}_\gamma$.

Of course, for every $(x, y) \notin \mathbf{L}_\gamma \cup \mathbf{U}_\gamma$, it is $l < \gamma < u$.

Figure 6 displays the sets $\mathbf{L}_\gamma, \mathbf{U}_\gamma$ when $\gamma = 0.4$. In the next result we determine in general the sets $\mathbf{L}_\gamma, \mathbf{U}_\gamma$.

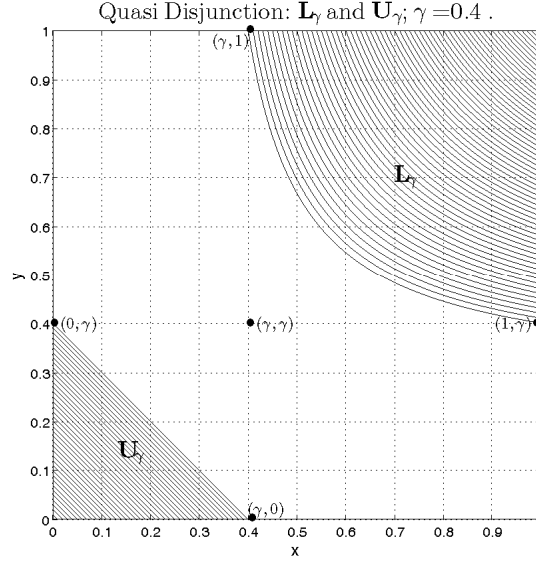


Figure 6: The sets $\mathbf{L}_\gamma, \mathbf{U}_\gamma$.

Theorem 13. Let be given the family $\mathcal{F}_n = \{E_1|H_1, \dots, E_n|H_n\}$, with the events $E_1, H_1, \dots, E_n, H_n$ logically independent. Moreover, for any given $\gamma \in [0, 1]$ let \mathbf{L}_γ (resp. \mathbf{U}_γ) be the set of the coherent assessments (p_1, p_2, \dots, p_n) on \mathcal{F}_n such that, for each $(p_1, p_2, \dots, p_n) \in \mathbf{L}_\gamma$ (resp. $(p_1, p_2, \dots, p_n) \in \mathbf{U}_\gamma$), one has $l \geq \gamma$ (resp. $u \leq \gamma$), where l is the lower bound (resp. u is the upper bound) of the coherent extensions $z = P[\mathcal{D}(\mathcal{F}_n)]$. We have

$$\begin{aligned} \mathbf{U}_\gamma &= \{(p_1, \dots, p_n) \in [0, 1]^n : p_1 + \dots + p_n \leq \gamma, \gamma < 1, \\ \mathbf{L}_\gamma &= \{(p_1, \dots, p_n) \in [0, 1]^n : \gamma \leq p_1 \leq 1, r_k \leq p_{k+1}, k = 1, \dots, n-1\}, \gamma > 0, \end{aligned} \quad (21)$$

where $r_k = \frac{\gamma l_k}{l_k - \gamma + \gamma l_k}$, $l_k = T_0^H(p_1, \dots, p_k)$, with $\mathbf{L}_0 = \mathbf{U}_1 = [0, 1]^n$.

Proof. Of course, $\mathbf{U}_1 = [0, 1]^n$, so that we can assume $\gamma < 1$. It must be $u_n = \min(p_1 + \dots + p_n, 1) \leq \gamma$, that is, as $\gamma < 1$, $p_1 + \dots + p_n \leq \gamma$. Hence: $\mathbf{U}_\gamma = \{(p_1, \dots, p_n) \in [0, 1]^n : p_1 + \dots + p_n \leq \gamma\}$.

We observe that \mathbf{U}_γ is a convex polyhedron with vertices the points

$$\begin{aligned} V_1 &= (\gamma, 0, \dots, 0), V_2 = (0, \gamma, 0, \dots, 0), \dots, \\ V_n &= (0, \dots, 0, \gamma), V_{n+1} = (0, 0, \dots, 0). \end{aligned}$$

Moreover, the convex hull of the vertices V_1, \dots, V_n is the subset of the points (p_1, \dots, p_n) of \mathbf{U}_γ such that $u_n = \gamma$, that is such that $p_1 + \dots + p_n = \gamma$.

Of course, $\mathbf{L}_0 = [0, 1]^n$, so that we can assume $\gamma > 0$. We recall that l_2, \dots, l_n are the lower bounds on $\mathcal{D}(\mathcal{F}_2), \dots, \mathcal{D}(\mathcal{F}_n)$ associated with (p_1, \dots, p_n) . Then, from the relations

$$\mathcal{D}(\mathcal{F}_{k+1}) = \mathcal{D}(\mathcal{D}(\mathcal{F}_k), E_{k+1}|H_{k+1}), \quad k = 2, \dots, n-1,$$

by applying (19) with $x = l_k, y = p_{k+1}$, we have that in order the inequality $l_{k+1} \geq \gamma$ be satisfied, it must be $l_k \geq \gamma, p_{k+1} \geq \gamma, k = 2, \dots, n-1$. Therefore

$$l_n \geq \gamma \implies p_1 \geq \gamma, \dots, p_n \geq \gamma, l_2 \geq \gamma, \dots, l_{n-1} \geq \gamma,$$

so that $\mathbf{L}_\gamma \subseteq [\gamma, 1]^n$. By iteratively applying (20), we obtain

$$\begin{aligned}
\gamma \leq p_1 \leq 1, \quad p_2 &\geq \frac{\gamma p_1}{p_1(1+\gamma) - \gamma} \implies l_2 \geq \gamma, \\
\gamma \leq l_2 \leq 1, \quad p_3 &\geq \frac{\gamma l_2}{l_2(1+\gamma) - \gamma} \implies l_3 \geq \gamma, \\
&\vdots \\
\gamma \leq l_{n-1} \leq 1, \quad p_n &\geq \frac{\gamma l_{n-1}}{l_{n-1}(1+\gamma) - \gamma} \implies l_n \geq \gamma.
\end{aligned}$$

Therefore, observing that $l_1 = p_1$, we have,

$$\mathbf{L}_\gamma = \{(p_1, \dots, p_n) \in [0, 1]^n : \gamma \leq p_1 \leq 1, \quad p_{k+1} \geq \frac{\gamma l_k}{l_k(1+\gamma) - \gamma}, \quad k = 1, \dots, n-1\}.$$

We observe that $\mathbf{L}_1 = \{(1, \dots, 1)\}$; moreover, for $p_1 = \dots = p_n = \gamma \in (0, 1)$, we obtain (by induction)

$$l_2 = \frac{\gamma}{2-\gamma} < \gamma, \quad l_3 = \frac{\gamma}{3-2\gamma} < \gamma, \quad l_4 = \frac{\gamma}{4-3\gamma} \dots, \quad l_n = \frac{\gamma}{n-(n-1)\gamma} < \gamma;$$

hence, for $\gamma \in (0, 1)$, \mathbf{L}_γ is a strict subset of $[0, \gamma]^n$. □

7. Biconditional Events, n -Conditional Events and Loop Rule

We now examine the quasi conjunction of $A|B$ and $B|A$, with A, B logically independent events. We have

$$C(A|B, B|A) = (AB \vee B^c) \wedge (BA \vee A^c) | (A \vee B) = AB | (A \vee B).$$

We observe that the conditional event $AB | (A \vee B)$ captures the notion of biconditional event² $A \dashv\vdash B$ considered by some authors as the “conjunction” between $A|B$ and $B|A$ and has the same truth table of the “defective biconditional” discussed in [29]; see also [27]. It can be easily verified that, for every pair $(x, y) \in [0, 1] \times [0, 1]$ the probability assessment (x, y) on $\{A|B, B|A\}$ is coherent. Given any coherent assessment (x, y) on $\{A|B, B|A\}$, the probability assessment $z = P(A \dashv\vdash B)$, is a coherent extension of (x, y) if and only if

$$z = \begin{cases} 0 & (x, y) = (0, 0), \\ \frac{xy}{x+y-xy} & (x, y) \neq (0, 0). \end{cases}$$

We can study the coherence of the assessment $\mathcal{P} = (x, y, z)$ on the family

$$\mathcal{F} = \{A|B, B|A, A \dashv\vdash B\} = \{A|B, B|A, AB | (A \vee B)\},$$

by the geometrical approach described in Section 2. In such a case, as the events of the family are not logically independent, the constituents generated by the family and contained in $A \vee B$ are: $C_1 = AB$, $C_2 = AB^c$, $C_3 = A^cB$. We distinguish two cases: (i) $(x, y) \neq (0, 0)$; (ii) $(x, y) = (0, 0)$.

- (i) If $(x, y) \neq (0, 0)$ the corresponding points Q_h 's are $Q_1 = (1, 1, 1)$, $Q_2 = (x, 0, 0)$, $Q_3 = (0, y, 0)$, and, in our case, the coherence of \mathcal{P} simply amounts to the geometrical condition $\mathcal{P} \in \mathcal{I}$, where \mathcal{I} is the triangle with vertices Q_1, Q_2, Q_3 . Based on the equation of the plane containing \mathcal{I} , we have that \mathcal{P} is coherent if and only if: $z = \frac{xy}{x+y-xy}$.
- (ii) If $(x, y) = (0, 0)$, then $Q_2 = Q_3 = (0, 0, 0)$ and the convex hull \mathcal{I} is the segment Q_1Q_2 . Then, $\mathcal{P} = (0, 0, z)$ is

²The representation of a biconditional event as a quasi conjunction was noticed in a private communication between A. Fugard and A. Gilio (January 2010).

coherent if and only if $z = 0$.

Then, the value z is a coherent extension of (x, y) if and only if

$$z = T_0^H(x, y) = \begin{cases} 0 & (x, y) = (0, 0), \\ \frac{xy}{x+y-xy} & (x, y) \neq (0, 0), \end{cases}$$

where $T_0^H(x, y)$ is the Hamacher t-norm, with parameter $\lambda = 0$, defined by formula (2). In agreement with Remark 2, we observe that

$$T_L(x, y) \leq T_0^H(x, y) \leq S_0^H(x, y).$$

7.1. Generalizing Biconditional Events: An Application to Loop rule

As shown before, given any (non impossible) events A_1, A_2 , the biconditional event associated with them is given by

$$A_1 \dashv\vdash A_2 = C(A_2|A_1, A_1|A_2) = A_1 A_2 | (A_1 \vee A_2).$$

The notion of biconditional event can be generalized by defining the n -conditional event associated with n (non impossible) events A_1, \dots, A_n as

$$A_1 \dashv\vdash A_2 \dashv\vdash \dots \dashv\vdash A_n = C(A_2|A_1, \dots, A_n|A_{n-1}, A_1|A_n).$$

Let C_0, C_1, \dots, C_m be the constituents generated by the conditional events $A_2|A_1, \dots, A_n|A_{n-1}, A_1|A_n$. We set $C_0 = A_1^c A_2^c \dots A_n^c$ and $C_1 = A_1 A_2 \dots A_n$; then, for each $h = 2, \dots, m$, it is $C_h = A_{i_1} \dots A_{i_r} A_{i_{r+1}}^c \dots A_{i_n}^c$, with $1 \leq r < n$. As it can be easily verified, the truth value of the n -conditional associated with C_h is true, or false, or void, according to whether $h = 1$, or $h > 1$, or $h = 0$; then it holds that

$$C(A_2|A_1, \dots, A_n|A_{n-1}, A_1|A_n) = A_1 \dots A_n | (A_1 \vee \dots \vee A_n).$$

In ([36]), where also the relationship with conditional objects ([22]) has been studied, the previous formula has been obtained by a suitable inductive reasoning, by showing that:

(i) $C(A_2|A_1, \dots, A_n|A_{n-1}) = (E_0 \vee \dots \vee E_{n-1})|(A_1 \vee \dots \vee A_{n-1})$,

where $E_0 = A_1 \dots A_n$, $E_1 = A_1^c A_2 \dots A_n$, \dots , $E_{n-2} = A_1^c \dots A_{n-2}^c A_{n-1} A_n$, $E_{n-1} = A_1^c \dots A_{n-1}^c$;

(ii) then

$$\begin{aligned} C(A_2|A_1, \dots, A_n|A_{n-1}, A_1|A_n) &= C[(E_1 \vee \dots \vee E_n)|(A_1 \vee \dots \vee A_{n-1}), A_1|A_n] = \\ &= A_1 \dots A_n | (A_1 \vee \dots \vee A_n). \end{aligned} \quad (22)$$

Of course, for any given derangement (a permutation with no fixed point) (i_1, i_2, \dots, i_n) of $(1, 2, \dots, n)$, we have

$$C(A_{i_1}|A_1, \dots, A_{i_{n-1}}|A_{n-1}, A_{i_n}|A_n) = C(A_2|A_1, \dots, A_n|A_{n-1}, A_1|A_n);$$

that is, the n -conditional $A_1 \dashv\vdash A_2 \dashv\vdash \dots \dashv\vdash A_n$ can be represented as the quasi conjunction of the conditional events $A_2|A_1, \dots, A_n|A_{n-1}, A_1|A_n$, or equivalently as the quasi conjunction of the conditional events $A_{i_1}|A_1, \dots, A_{i_{n-1}}|A_{n-1}, A_{i_n}|A_n$. In particular for $(i_1, i_2, \dots, i_n) = (n, 1, 2, \dots, n-1)$ we have

$$C(A_1|A_2, \dots, A_{n-1}|A_n, A_n|A_1) = C(A_2|A_1, \dots, A_n|A_{n-1}, A_1|A_n). \quad (23)$$

As a consequence, we can immediately obtain the probabilistic interpretation of *Loop* rule ([54]). Given n logically independent events A_1, A_2, \dots, A_n , Loop rule is the following one:

$$A_1 \vdash A_2, A_2 \vdash A_3, \dots, A_n \vdash A_1 \implies A_1 \vdash A_n. \quad (24)$$

In [54] it has also been proved that, for every $i, j = 1, 2, \dots, n$, it holds that

$$A_1 \vdash A_2, A_2 \vdash A_3, \dots, A_n \vdash A_1 \implies A_i \vdash A_j. \quad (25)$$

7.2. Probabilistic Aspects on Loop Rule

In our probabilistic approach, formula (25), which generalizes formula (24), can be obtained by the following steps:

- given any p-consistent family of conditional events \mathcal{F} , from Corollary 1 it holds that \mathcal{F} p-entails $C(\mathcal{F})$;
- defining $\mathcal{F} = \{A_2|A_1, \dots, A_n|A_{n-1}, A_1|A_n\}$, it can be checked that \mathcal{F} is p-consistent; then, for every $i, j = 1, 2, \dots, n$, by (22) $C(\mathcal{F}) \subseteq A_i|A_j$; hence $C(\mathcal{F})$ p-entails $A_i|A_j$; moreover, \mathcal{F} p-entails $C(\mathcal{F})$ and then \mathcal{F} p-entails $A_i|A_j$.

Remark 7. By Definition 5 and formulas (23) and (25), for any given derangement (i_1, i_2, \dots, i_n) of $(1, 2, \dots, n)$, we obtain the following inference rule (*Generalized Loop*)

$$\{A_2|A_1, \dots, A_n|A_{n-1}, A_1|A_n\} \begin{matrix} \Rightarrow_p \\ \Leftarrow_p \end{matrix} \{A_{i_1}|A_1, \dots, A_{i_{n-1}}|A_{n-1}, A_{i_n}|A_n\} \quad (26)$$

The Loop rule has been studied by a direct probabilistic reasoning in [36], by exploiting a suitable probabilistic condition named *Császár's condition*, studied in the framework of an axiomatic approach to probability in [20]. This condition in a particular case reduces to the third axiom of conditional probabilities. A numerical inference rule named *generalized Bayes theorem*, connected with Császár's condition and with Loop rule, has been studied in [4]; see also [5, 23]. Below, we reconsider an example introduced in [36] to illustrate the generalized Loop rule and p-entailment of n -conditionals.

Example 4. Five friends, Linda, Janet, Steve, George, and Peter, have been invited to a party. We define the events: $A_1 = \text{"Linda goes to the party"}$, \dots , $A_5 = \text{"Peter goes to the party"}$; moreover, we assume that A_1, \dots, A_5 are logically independent. We consider the following knowledge base: {"if Linda goes to the party, then Janet will do the same", \dots , "if George goes to the party, then Peter will do the same", "if Peter goes to the party, then Linda will do the same"}. Then, for the associated (p-consistent) family of conditional events $\mathcal{F} = \{A_2|A_1, \dots, A_5|A_4, A_1|A_5\}$, we have

$$C(\mathcal{F}) = A_1 A_2 \cdots A_5 | (A_1 \vee A_2 \vee \cdots \vee A_5) = A_1 \dashv\vdash A_2 \dashv\vdash \cdots \dashv\vdash A_5.$$

By generalized Loop rule, for every derangement (i_1, \dots, i_5) of $(1, \dots, 5)$, it holds that

$$\{A_2|A_1, \dots, A_5|A_4, A_1|A_5\} \begin{matrix} \Rightarrow_p \\ \Leftarrow_p \end{matrix} \{A_{i_1}|A_1, \dots, A_{i_4}|A_4, A_{i_5}|A_5\}.$$

For any given subset $\{B_1, \dots, B_n\} \subset \{A_1, \dots, A_5\}$, $n = 2, 3, 4$, we have $B_1 \dashv\vdash \cdots \dashv\vdash B_n = B_1 \cdots B_n | (B_1 \vee \cdots \vee B_n)$. This n -conditional is associated with the conditional assertion "if at least one of n given friends among Linda, Janet, Steve, George, and Peter, goes to the party, then all n friends will go to the party". We have

$$A_1 \cdots A_5 | (A_1 \vee \cdots \vee A_5) \subseteq B_1 \cdots B_n | (B_1 \vee \cdots \vee B_n);$$

therefore $A_1 \dashv\vdash \cdots \dashv\vdash A_5$ p-entails $B_1 \dashv\vdash \cdots \dashv\vdash B_n$. Finally, as \mathcal{F} p-entails $C(\mathcal{F})$, we have that for every subset $\{B_1, \dots, B_n\}$, $n = 2, 3, 4$, the family \mathcal{F} p-entails the n -conditional $B_1 \dashv\vdash \cdots \dashv\vdash B_n$.

8. Conclusions

In this paper we have examined probabilistic concepts connected with the inference rules Quasi And, Quasi Or, Or, and generalized Loop. These are linked with Adams' probabilistic analysis of conditionals, and play an important role in applications to nonmonotonic reasoning, to the psychology of uncertain reasoning and to semantic web. We have considered, in a coherence-based setting, the extensions of a given probability assessment on n conditional events to their quasi conjunction and quasi disjunction, by also examining some cases of logical dependencies. In our probabilistic analysis we have shown that the lower and upper probability bounds computed in the different cases coincide with some well known t-norms and t-conorms: minimum, product, Lukasiewicz and Hamacher t-norms, and their dual t-conorms. We have shown that, for the Or rule, the quasi conjunction and quasi disjunction of the premises are equal. Moreover, they coincide with the conclusion of the rule. We have identified the relationships among coherence, inclusion relation and p-entailment. Finally, we have considered biconditional events and we have introduced

the notion of n -conditional event, by obtaining a probabilistic interpretation for a generalized Loop rule. In Appendix C we give explicit expressions for the Hamacher t -norm and t -conorm in the unitary hypercube $[0, 1]^k$. As a "take home message", the results obtained in our coherence-based probabilistic approach can be exploited in all researches in nonmonotonic reasoning, as made for instance in [38, 51, 60, 61]. Future work should deepen the theoretical aspects and applications which connect conditional probability with t -norms and t -conorms, in relation to inference patterns in nonmonotonic reasoning. In particular, the representation of probability bounds for the conditional conclusions of some inference patterns involving conditionals in terms of t -norms and t -conorms is a topic that could be expanded. Finally, a relevant topic for further research concerns the study of more general definitions for the logical operations of conjunction and disjunction among conditionals. Such new logical operations should be defined in a way such that the usual probabilistic properties be preserved. Some results on this topic have been given in [43].

Acknowledgements. The authors thank the editors and four anonymous reviewers for their valuable criticisms and comments, which were helpful in improving the paper. The authors also acknowledge Frank Lad for his useful suggestions regarding the linguistic quality of some parts of the manuscript.

References

- [1] E.W. Adams, *The Logic of Conditionals*, Reidel, Dordrecht, 1975.
- [2] M.M. Ali, N.N. Mikhail, M. Haq, A class of bivariate distributions including the bivariate logistic, *J. Multivariate Anal.* 8 (1978) 405–412.
- [3] C. Alsina, M.J. Frank, B. Schweizer, *Associative Functions: Triangular Norms and Copulas*, World Scientific, 2006.
- [4] S. Amarger, D. Dubois, H. Prade, Constraint Propagation with Imprecise Conditional Probabilities, in: *Proc. of the 7th Conf. on Uncertainty in Artificial Intelligence (UAI-91)*, Morgan Kaufmann, 1991, pp. 26–34.
- [5] S. Amarger, D. Dubois, H. Prade, Handling imprecisely-known conditional probabilities, in: D.J. Hand (Ed.), *AI and Computer Power: The Impact on Statistics*, Chapman & Hall, 1994, pp. 63–97.
- [6] S. Benferhat, D. Dubois, H. Prade, Nonmonotonic Reasoning, Conditional Objects and Possibility Theory, *Artif. Intell.* 92 (1997) 259–276.
- [7] V. Biazzo, A. Gilio, A generalization of the fundamental theorem of de Finetti for imprecise conditional probability assessments, *Internat. J. Approx. Reason.* 24 (2000) 251–272.
- [8] V. Biazzo, A. Gilio, T. Lukasiewicz, G. Sanfilippo, Probabilistic logic under coherence, model-theoretic probabilistic logic, and default reasoning in System P, *J. Appl. Non-Classical Logics* 12 (2002) 189–213.
- [9] V. Biazzo, A. Gilio, T. Lukasiewicz, G. Sanfilippo, Probabilistic logic under coherence: complexity and algorithms., *Ann. Math. Artif. Intell.* 45 (2005) 35–81.
- [10] V. Biazzo, A. Gilio, G. Sanfilippo, Coherence checking and propagation of lower probability bounds, *Soft Computing* 7 (2003) 310–320.
- [11] V. Biazzo, A. Gilio, G. Sanfilippo, On the Checking of G-Coherence of Conditional Probability Bounds., *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems* 11, Suppl.2 (2003) 75–104.
- [12] V. Biazzo, A. Gilio, G. Sanfilippo, Coherent Conditional Previsions and Proper Scoring Rules, in: S. Greco, B. Bouchon-Meunier, G. Coletti, M. Fedrizzi, B. Matarazzo, R.R. Yager (Eds.), *Advances in Computational Intelligence*, volume 300 of *CCIS*, Springer, 2012, pp. 146–156.
- [13] A. Brozzi, A. Capotorti, B. Vantaggi, Incoherence correction strategies in statistical matching, *Int. J. Approx. Reason.* 53 (2012) 1124–1136.
- [14] A. Capotorti, F. Lad, G. Sanfilippo, Reassessing Accuracy Rates of Median Decisions, *The American Statistician* 61 (2007) 132–138.
- [15] A. Capotorti, G. Regoli, F. Vattari, Correction of incoherent conditional probability assessments, *Int. J. Approx. Reason.* 51 (2010) 718–727.
- [16] D. Ciucci, D. Dubois, Relationships between Connectives in Three-Valued Logics, in: S. Greco, B. Bouchon-Meunier, G. Coletti, M. Fedrizzi, B. Matarazzo, R. Yager (Eds.), *Advances on Computational Intelligence*, volume 297 of *CCIS*, Springer, 2012, pp. 633–642.
- [17] G. Coletti, O. Gervasi, S. Tasso, B. Vantaggi, Generalized Bayesian inference in a fuzzy context: From theory to a virtual reality application, *Comput. Statist. Data Anal.* 56 (2012) 967–980.
- [18] G. Coletti, R. Scozzafava, Probabilistic logic in a coherent setting, volume 15 of *Trends in logics*, Kluwer, Dordrecht, 2002.
- [19] G. Coletti, R. Scozzafava, B. Vantaggi, Inferential processes leading to possibility and necessity, *Information Sciences* (2012). Doi 10.1016/j.ins.2012.10.034.
- [20] A. Császár, Sur la structure des espaces de probabilité conditionnelle, *Acta Mathematica Academiae Scientiarum Hungarica* 6 (1955) 337–361.
- [21] J. Dombi, Towards a General Class of Operators for Fuzzy Systems, *IEEE Trans. Fuzzy Syst.* 16 (2008) 477–484.
- [22] D. Dubois, H. Prade, Conditional objects as nonmonotonic consequence relationships, *IEEE Trans. Syst., Man, Cybern.* 24 (1994) 1724–1740.
- [23] D. Dubois, H. Prade, J.M. Toucas, Inference with imprecise numerical quantifiers, in: Z. Ras, M. Zemankova (Eds.), *Intelligent Systems: State of the Art and Future Directions*, Ellis Horwood Ltd., 1990, pp. 57–72.
- [24] B. de Finetti, Does it make sense to speak of 'good probability appraisers'?, in: I.J. Good (Ed.), *The scientist speculates: an anthology of partly-baked ideas*, Heinemann, London, 1962, pp. 357–364.
- [25] B. de Finetti, Probabilità composte e teoria delle decisioni, *Rendiconti di Matematica* 23 (1964) 128–134.
- [26] B. de Finetti, *Teoria delle probabilità*, Ed. Einaudi, 2 voll., Torino, 1970.
- [27] A.J.B. Fugard, N. Pfeifer, B. Mayerhofer, G.D. Kleiter, How people interpret conditionals: Shifts toward the conditional event, *J. Exp. Psychol. Learn. Mem. Cogn.* 37 (2011) 635–648.
- [28] D. Gale, *The theory of linear economic models*, McGraw-Hill, NY, 1960.
- [29] C. Gauffroy, P. Barrouillet, Heuristic and analytic processes in mental models for conditionals: An integrative developmental theory, *Developmental Review* 29 (2009) 249–282.

- [30] A. Gilio, Criterio di penalizzazione e condizioni di coerenza nella valutazione soggettiva della probabilità, *Boll. Un. Mat. Ital.* 4-B (1990) 645–660.
- [31] A. Gilio, C_0 -Coherence and Extension of Conditional Probabilities, in: J.M. Bernardo, J.O. Berger, A.P. Dawid, A.F.M. Smith (Eds.), *Bayesian Statistics 4*, Oxford University Press, 1992, pp. 633–640.
- [32] A. Gilio, Probabilistic Consistency of Knowledge Bases in Inference Systems, in: M. Clarke, R. Kruse, S. Moral (Eds.), *ECSQARU*, volume 747 of *LNCS*, Springer, 1993, pp. 160–167.
- [33] A. Gilio, Algorithms for precise and imprecise conditional probability assessments, in: G. Coletti, D. Dubois, R. Scozzafava (Eds.), *Mathematical Models for Handling Partial Knowledge in Artificial Intelligence*, Plenum Press, New York, 1995, pp. 231–254.
- [34] A. Gilio, Algorithms for conditional probability assessments, in: D.A. Berry, K.M. Chaloner, J.K. Geweke (Eds.), *Bayesian Analysis in Statistics and Econometrics: Essays in Honor of Arnold Zellner*, John Wiley, NY, 1996, pp. 29–39.
- [35] A. Gilio, Probabilistic Reasoning Under Coherence in System P, *Ann. Math. Artif. Intell.* 34 (2002) 5–34.
- [36] A. Gilio, On Császár's Condition in Nonmonotonic Reasoning, in: 10th International Workshop on Non-Monotonic Reasoning. Special Session: Uncertainty Frameworks in Non-Monotonic Reasoning, Whistler BC, Canada, June 6–8, 2004. <http://events.pims.math.ca/science/2004/NMR/uf.html>.
- [37] A. Gilio, Generalizing inference rules in a coherence-based probabilistic default reasoning, *Int. J. Approx. Reasoning* 53 (2012) 413–434.
- [38] A. Gilio, D. Over, The psychology of inferring conditionals from disjunctions: A probabilistic study, *Journal of Mathematical Psychology* 56 (2012) 118–131.
- [39] A. Gilio, G. Sanfilippo, Quasi Conjunction and p-entailment in Nonmonotonic Reasoning, in: C. Borgelt, G. Rodríguez, W. Trutschnig, M.A. Lubiano, M.Á. Gil, P. Grzegorzewski, O. Hryniewicz (Eds.), *Combining Soft Computing and Statistical Methods in Data Analysis*, volume 77 of *AISC*, Springer, 2010, pp. 321–328.
- [40] A. Gilio, G. Sanfilippo, Coherent conditional probabilities and proper scoring rules, in: F. Coolen, G. de Cooman, T. Fetz, M. Oberguggenberger (Eds.), *ISIPTA'11: Proceedings of the Seventh International Symposium on Imprecise Probability: Theories and Applications*, SIPTA, Innsbruck, 2011, pp. 189–198.
- [41] A. Gilio, G. Sanfilippo, Quasi conjunction and inclusion relation in probabilistic default reasoning, in: W. Liu (Ed.), *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, volume 6717 of *LNCS*, Springer, 2011, pp. 497–508.
- [42] A. Gilio, G. Sanfilippo, Probabilistic entailment in the setting of coherence: The role of quasi conjunction and inclusion relation, *Int. J. Approx. Reasoning* 54 (2013) 513–525.
- [43] A. Gilio, G. Sanfilippo, Conjunction, Disjunction and Iterated Conditioning of Conditional Events, in: R. Kruse, M.R. Berthold, C. Moewes, M.A. Gil, P. Grzegorzewski, O. Hryniewicz (Eds.), *Synergies of Soft Computing and Statistics for Intelligent Data Analysis*, volume 190 of *Advances in Intelligent Systems and Computing*, Springer, 2013, pp. 399–407.
- [44] L. Godo, E. Marchioni, Coherent Conditional Probability in a Fuzzy Logic Setting, *Logic Journal of the IGPL* 14 (2006) 457–481.
- [45] I.R. Goodman, H.T. Nguyen, Conditional Objects and the Modeling of Uncertainties, in: M.M. Gupta, T. Yamakawa (Eds.), *Fuzzy Computing*, North-Holland, 1988, pp. 119–138.
- [46] M. Grabisch, J.L. Marichal, R. Mesiar, E. Pap, *Aggregation functions*, Cambridge University Press, 2009.
- [47] M. Grabisch, J.L. Marichal, R. Mesiar, E. Pap, Aggregation functions: Construction methods, conjunctive, disjunctive and mixed classes, *Information Sciences* 181 (2011) 23–43.
- [48] M. Grabisch, J.L. Marichal, R. Mesiar, E. Pap, Aggregation functions: Means, *Information Sciences* 181 (2011) 1–22.
- [49] H. Hamacher, Über logische Aggregationen nicht-binär explizierter Entscheidungskriterien, Rita G. Fischer Verlag, 1978.
- [50] G. Kern-Isberner, Conditionals in Nonmonotonic Reasoning and Belief Revision, volume 2087 of *LNCS*, Springer, 2001.
- [51] G. Kleiter, Ockham's Razor in Probability Logic, in: R. Kruse, M.R. Berthold, C. Moewes, M.A. Gil, P. Grzegorzewski, O. Hryniewicz (Eds.), *Synergies of Soft Computing and Statistics for Intelligent Data Analysis*, volume 190 of *Advances in Intelligent Systems and Computing*, Springer, 2013, pp. 409–417.
- [52] E.P. Klement, R. Mesiar, E. Pap, *Triangular Norms*, Springer, 2000.
- [53] E.P. Klement, R. Mesiar, E. Pap, Triangular norms: basic notions and properties, in: *Logical, algebraic, analytic and probabilistic aspects of triangular norms*, Elsevier, 2005, pp. 17–60.
- [54] S. Kraus, D. Lehmann, M. Magidor, Nonmonotonic reasoning, preferential models and cumulative logics, *Artif. Intell.* 44 (1990) 167–207.
- [55] F. Lad, *Operational Subjective Statistical Methods*, Wiley, 1996.
- [56] F. Lad, G. Sanfilippo, G. Agró, Completing the logarithmic scoring rule for assessing probability distributions, *AIP Conference Proceedings* 1490 (2012) 13–30.
- [57] T. Lukasiewicz, U. Straccia, Managing uncertainty and vagueness in description logics for the Semantic Web, *Journal of Web Semantics* 6 (2008) 291–308.
- [58] K. Menger, *Statistical Metrics*, *Proc Natl Acad Sci U S A* 28 (1942) 535–537.
- [59] R.B. Nelsen, An Introduction to Copulas, volume 139 of *Lecture Notes in Statistics*, Springer, 1999.
- [60] N. Pfeifer, G.D. Kleiter, Inference in conditional probability logic, *Kybernetika* 42 (2006) 391–404.
- [61] N. Pfeifer, G.D. Kleiter, Framing human inference by coherence based probability logic, *Journal of Applied Logic* 7 (2009) 206–217.
- [62] G. Sanfilippo, From imprecise probability assessments to conditional probabilities with quasi additive classes of conditioning events, in: *Proc. of the Twenty-Eighth Conference on Uncertainty in Artificial Intelligence (UAI-12)*, AUAI Press, Corvallis, Oregon, 2012, pp. 736–745.
- [63] B. Schweizer, A. Sklar, Associative functions and statistical triangle inequalities, *Publ. Math.* 8 (1961) 169–186.
- [64] R. Scozzafava, B. Vantaggi, Fuzzy inclusion and similarity through coherent conditional probability, *Fuzzy Sets and Systems* 160 (2009) 292–305.
- [65] M. Thimm, G. Kern-Isberner, J. Fisseler, Relational Probabilistic Conditional Reasoning at Maximum Entropy, in: W. Liu (Ed.), *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, volume 6717 of *LNCS*, Springer, 2011, pp. 447–458.
- [66] R.D. Tweney, M.E. Doherty, G.D. Kleiter, The pseudodiagnosticity trap: Should participants consider alternative hypotheses?, *Thinking & Reasoning* 16 (2010) 332–345.

Appendix A. t-norms and t-conorms.

We recall below the notions of t-norm and t-conorm (see [48, 52, 53]).

Definition 9. A *t-norm* is a function $T : [0, 1]^2 \longrightarrow [0, 1]$ which satisfies, for all $x, y, z \in [0, 1]$, the following four axioms:

$$\begin{aligned} (T1) \quad & T(x, y) = T(y, x), & (\text{commutativity}) \\ (T2) \quad & T(x, T(y, z)) = T(T(x, y), z), & (\text{associativity}) \\ (T3) \quad & T(x, y) \leq T(x, z) \text{ whenever } y \leq z, & (\text{monotonicity}) \\ (T4) \quad & T(x, 1) = x. & (\text{boundary condition}) \end{aligned}$$

We recall below some basic t-norms, namely, the *minimum* T_M (which is the greatest t-norm), the product T_P , the *Lukasiewicz* t-norm T_L :

$$T_M(x, y) = \min(x, y), \quad T_P(x, y) = x \cdot y, \quad T_L(x, y) = \max(x + y - 1, 0).$$

We also recall that the *Hamacher* t-norm T_λ^H , with parameter $\lambda \in [0, \infty]$, is

$$T_\lambda^H(x, y) = \begin{cases} T_D(x, y), & \lambda = \infty, \\ 0, & \lambda = 0 \text{ and } (x, y) = (0, 0) \\ \frac{xy}{\lambda + (1-\lambda)(x+y-xy)}, & \text{otherwise,} \end{cases} \quad (\text{A.1})$$

where the t-norm $T_D(x, y)$ (*drastic product*) is defined as

$$T_D(x, y) = \begin{cases} 0, & (x, y) \in [0, 1)^2, \\ \min(x, y), & \text{otherwise.} \end{cases}$$

In particular, the Hamacher t-norm T_1^H is the product t-norm T_P .

Definition 10. A *t-conorm* is a function $S : [0, 1]^2 \longrightarrow [0, 1]$ which satisfies, for all $x, y, z \in [0, 1]$, (T1) – (T3) and

$$(S4) \quad S(x, 0) = x. \quad (\text{boundary condition})$$

T-conorms can be equivalently introduced as dual operations of t-norms. A function $S : [0, 1]^2 \longrightarrow [0, 1]$, is a t-conorm if and only if there exists a t-norm T such that for all $(x, y) \in [0, 1]^2$ either one of the two equalities holds: $S(x, y) = 1 - T(1 - x, 1 - y)$, $T(x, y) = 1 - S(1 - x, 1 - y)$. Then, the dual t-conorm of T_M is the *maximum* S_M , i.e. $S_M(x, y) = \max(x, y)$. The dual t-conorm of T_P is the *probabilistic sum* S_P , i.e.

$$S_P(x, y) = 1 - (1 - x)(1 - y) = x + y - x \cdot y.$$

The Lukasiewicz t-conorm, which is the dual t-conorm of T_L , is

$$S_L(x, y) = \min(x + y, 1).$$

Moreover, the Hamacher t-conorm S_λ^H with parameter $\lambda \in [0, \infty]$, which is the dual t-conorm of T_λ^H , is

$$S_\lambda^H(x, y) = \begin{cases} S_D(x, y), & \lambda = \infty, \\ 1, & \lambda = 0 \text{ and } x = y = 1, \\ \frac{x+y-xy-(1-\lambda)xy}{1-(1-\lambda)xy}, & \text{otherwise,} \end{cases} \quad (\text{A.2})$$

where the t-conorm $S_D(x, y)$ (*drastic sum*) is defined as

$$S_D(x, y) = \begin{cases} 1, & (x, y) \in (0, 1]^2, \\ \max(x, y), & \text{otherwise.} \end{cases}$$

In particular, the Hamacher t-conorm S_1^H is the probabilistic sum S_P .

Appendix B. t-norms and t-conorms in $[0, 1]^k$.

We recall that since t-norms and t-conorms are associative they can be easily extended in a unique way to a k -ary operation for arbitrary integer $k \geq 2$ by induction (see [46, 48, 53]). Let T be a t-norm (introduced as a binary operator), for any integer $k \geq 2$ the extension of T is defined as

$$T(p_1, p_2, \dots, p_k) = \begin{cases} T(T(p_1, \dots, p_{k-1}), p_k), & \text{if } k > 2, \\ T(p_1, p_2), & \text{if } k = 2. \end{cases}$$

Let S be a t-conorm (introduced as a binary operator), for any integer $k \in \mathbb{N} \cup \{0\}$ the extension of S is defined as

$$S(p_1, p_2, \dots, p_k) = \begin{cases} S(S(p_1, \dots, p_{k-1}), p_k), & \text{if } k > 2, \\ S(p_1, p_2), & \text{if } k = 2. \end{cases}$$

If (T, S) is a pair of mutually dual t-norms and t-conorms, then

$$\begin{aligned} S(p_1, \dots, p_k) &= 1 - T(1 - p_1, \dots, 1 - p_k), \\ T(p_1, \dots, p_k) &= 1 - S(1 - p_1, \dots, 1 - p_k). \end{aligned}$$

Finally, we recall that

$$\begin{aligned} T_M(p_1, \dots, p_k) &= \min(p_1, \dots, p_k), \quad S_M(p_1, \dots, p_k) = \max(p_1, \dots, p_k), \\ T_p(p_1, \dots, p_k) &= p_1 \cdots p_k, \quad S_p(p_1, \dots, p_k) = 1 - (1 - p_1) \cdots (1 - p_k), \\ T_L(p_1, p_2, \dots, p_k) &= \max(p_1 + p_2 + \dots + p_k - (k - 1), 0), \\ S_L(p_1, p_2, \dots, p_k) &= \min(p_1 + p_2 + \dots + p_k, 1). \end{aligned}$$

Appendix C. Hamacher t-norm and t-conorm in $[0, 1]^k$

In this appendix, by using the notion of additive generator, we give self contained constructions of the extensions of the Hamacher t-norm and t-conorm with $\lambda = 0$ to $[0, 1]^k$.

We recall the notion of an additive generator (if any) of a t-norm ([52, 53]).

Definition 11. An additive generator $t : [0, 1] \rightarrow [0, \infty]$ of a t-norm T is a strictly decreasing function which is also right continuous in 0 and satisfies $t(1) = 0$, such that for all $(x, y) \in [0, 1]^2$ we have

$$t(x) + t(y) \in \text{Ran}(t) \cup [t(0), \infty] \quad \text{and} \quad T(x, y) = t^{-1}(t(x) + t(y)),$$

where $\text{Ran}(t) = \{t(x) : x \in [0, 1]\}$ and t^{-1} is the pseudo inverse of t .

If t is an additive generator of some t-norm T , then we have

$$T(p_1, p_2, \dots, p_k) = t^{-1}(t(p_1) + t(p_2) + \dots + t(p_k)). \quad (\text{C.1})$$

We first observe that: if $(x = 0, y = 0)$, then $T_0^H(x, y) = 0$; if $(x = 0, y > 0)$ or $(x > 0, y = 0)$, then $T_0^H(x, y) = \frac{xy}{x+y-xy} = 0$; if $(x > 0, y > 0)$, then

$$T_0^H(x, y) = \frac{xy}{x+y-xy} = \frac{xy}{x(1-y)+y(1-x)+xy} = \frac{1}{\frac{1-x}{x} + \frac{1-y}{y} + 1} > 0.$$

Thus, T_0^H can be equivalently redefined as

$$T_0^H(x, y) = \begin{cases} 0, & (x = 0) \vee (y = 0), \\ \frac{1}{\frac{1-x}{x} + \frac{1-y}{y} + 1}, & (x \neq 0) \wedge (y \neq 0). \end{cases} \quad (\text{C.2})$$

We have (see also [46, 52])

Proposition 3. Let T_0^H be the Hamacher t-norm with $\lambda = 0$. Given an integer $k \geq 2$, the extension of T_0^H to $[0, 1]^k$ is

$$T_0^H(p_1, p_2, \dots, p_k) = \begin{cases} 0, & p_i = 0 \text{ for at least one } i, \\ \frac{1}{\sum_{i=1}^k \frac{1-p_i}{p_i} + 1}, & p_i > 0 \text{ for } i = 1, \dots, k. \end{cases} \quad (\text{C.3})$$

Proof. We observe that, considering the function $t : [0, 1] \rightarrow [0, +\infty]$ defined as $t(x) = \frac{1-x}{x}$, with the convention that $t(0) = \lim_{x \rightarrow 0^+} \frac{1-x}{x} = +\infty$, it holds $t^{-1}(s) = \frac{1}{1+s}$, if $s \in [0, \infty]$, with $t^{-1}(+\infty) = 0$. Then, by applying the conventions $\frac{1}{\infty} = 0$, $\frac{1}{0} = \infty$ and recalling (C.2), for every $(x, y) \in [0, 1]^2$ we have

$$t^{-1}(t(x) + t(y)) = \frac{1}{1 + \frac{1-x}{x} + \frac{1-y}{y}} = T_0^H(x, y);$$

As the function $t(x) = \frac{1-x}{x}$ is the additive generator of T_0^H , we have

$$T_0^H(p_1, p_2, \dots, p_k) = t^{-1}(\sum_{i=1}^k t(p_i)) = t^{-1}(\sum_{i=1}^k \frac{1-p_i}{p_i}) = \frac{1}{1 + \sum_{i=1}^k \frac{1-p_i}{p_i}}.$$

□

Now, we observe that: if $x = 1$ and $y = 1$, then $S_0^H(x, y) = 1$; if $(x = 1, y < 1)$ or $(x < 1, y = 1)$, then $S_0^H(x, y) = \frac{x+y-2xy}{1-xy} = 1$; if $x < 1$ and $y < 1$ we have

$$\begin{aligned} S_0^H(x, y) &= \frac{x+y-2xy}{1-xy} = \frac{x(1-y)+y(1-x)}{x(1-y)+y(1-x)+(1-x)(1-y)} = \\ &= \frac{\frac{x}{(1-x)}(1-x)(1-y) + \frac{y}{(1-y)}(1-x)(1-y)}{\frac{x}{(1-x)}(1-x)(1-y) + \frac{y}{(1-y)}(1-x)(1-y) + (1-x)(1-y)} = \frac{\frac{x}{(1-x)} + \frac{y}{(1-y)}}{\frac{x}{(1-x)} + \frac{y}{(1-y)} + 1} < 1. \end{aligned}$$

Thus, the Hamacher t-conorm $S_0^H : [0, 1]^2 \rightarrow [0, 1]$ can be equivalently redefined as

$$S_0^H(x, y) = \begin{cases} 1, & (x = 1) \vee (y = 1), \\ \frac{\frac{x}{(1-x)} + \frac{y}{(1-y)}}{\frac{x}{(1-x)} + \frac{y}{(1-y)} + 1}, & (x < 1) \wedge (y < 1). \end{cases} \quad (\text{C.4})$$

By observing that $S(p_1, p_2, \dots, p_k) = 1 - T(1 - p_1, 1 - p_2, \dots, 1 - p_k)$, it immediately follows

Proposition 4. Let S_0^H be the Hamacher t-conorm with $\lambda = 0$. Given an integer $k \geq 2$, for any vector $(p_1, p_2, \dots, p_k) \in [0, 1]^k$ it holds that

$$S_0^H(p_1, p_2, \dots, p_k) = \begin{cases} 1, & p_i = 1 \text{ for at least one } i, \\ \frac{\sum_{i=1}^k \frac{p_i}{1-p_i}}{\sum_{i=1}^k \frac{p_i}{1-p_i} + 1}, & p_i < 1 \text{ for } i = 1, \dots, k. \end{cases} \quad (\text{C.5})$$

Remark 8. We observe that the Hamacher t-norm T_0^H and Hamacher t-conorm S_0^H coincide, respectively for $\alpha = 1$ and $\alpha = -1$, with the Dombi operator defined as ([21]):

$$o(p_1, \dots, p_k) = \frac{1}{1 + \left(\sum_{i=1}^k \left(\frac{1-p_i}{p_i} \right)^\alpha \right)^{\frac{1}{\alpha}}}.$$